

# Morse theory methods for quasi-linear elliptic systems of higher order \*

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## Abstract

We develop the local Morse theory for a class of non-twice continuously differentiable functionals on Hilbert spaces, including a new generalization of the Gromoll-Meyer's splitting theorem and a weaker Marino-Prodi perturbation type result. With them some critical point theorems and famous bifurcation theorems are generalized. Then we show that these are applicable to studies of quasi-linear elliptic equations and systems of higher order given by multi-dimensional variational problems as in (1.3).

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>I</b>	<b>Abstract theories</b>	<b>7</b>
<b>2</b>	<b>Local Morse theory for a class of non-<math>C^2</math> functionals</b>	<b>8</b>
2.1	Statements of main results . . . . .	8
2.2	Lemmas . . . . .	12
2.3	Proof of Theorem 2.1 . . . . .	13
2.4	An implicit function theorem for a family of potential operators . . . . .	15
2.5	Parameterized splitting and shifting theorems . . . . .	26
2.6	Splitting and shifting theorems around critical orbits . . . . .	34
2.7	Proof of Theorem 2.6 . . . . .	42

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<b>3</b>	<b>Bifurcations for potential operators</b>	<b>45</b>
3.1	Generalizations of a bifurcation theorem by Chow and Lauterbach . . . . .	46
3.2	Generalizations of Rabinowitz bifurcation theorem . . . . .	47
3.3	Bifurcation for equivariant problems . . . . .	56
<b>II</b>	<b>Applications to quasi-linear elliptic systems of higher order</b>	<b>68</b>
<b>4</b>	<b>Fundamental analytic properties for functionals <math>\mathcal{F}</math> and <math>\mathfrak{F}</math></b>	<b>68</b>
4.1	Results and preliminaries . . . . .	68
4.2	Proof for A) of Theorem 4.1 . . . . .	72
4.3	Proof for B) of Theorem 4.1 . . . . .	75
4.4	Proof for C) of Theorem 4.1 . . . . .	81
4.5	Proof for D) of Theorem 4.1 . . . . .	85
<b>5</b>	<b>(PS)- and (C)-conditions</b>	<b>87</b>
<b>6</b>	<b>Morse inequalities</b>	<b>90</b>
<b>7</b>	<b>Bifurcations for Quasi-linear elliptic systems</b>	<b>92</b>
<b>8</b>	<b>Concluding remarks</b>	<b>99</b>
<b>A</b>	<b>Proof of Proposition 4.3</b>	<b>99</b>

## 1 Introduction

Since Palais and Smale [53, 55, 61] generalized finite-dimensional Morse theory in [51] to nondegenerate  $C^2$  functionals on infinite dimensional Hilbert manifolds and used it to study multiplicity of solutions for semilinear elliptic boundary value problems, via Gromoll and Meyer [31], Marino and Prodi [49] and many other people's effort, such a direction has very successful developments, see a few of nice books [3, 12, 13, 14, 50, 52, 56, 58, 70] and references therein for details. The Morse theory for functionals on an infinite dimensional Hilbert space  $H$  have two main aspects: Morse relations related critical groups to Betti numbers of underlying spaces (global), computation of critical groups (local). The basic tool for the latter, Gromoll-Meyer's generalized Morse lemma (or splitting theorem) in [31], was only generalized to  $C^2$  functionals on Hilbert spaces ([13, 50]) until author's recent work [39, 40]. Because of this, most of applications of the theory to differential equations are restricted to semi-linear elliptic equations and Hamiltonian systems [13, 50, 52]. Skrypnik [58] established Morse inequalities for the functional (1.8) with  $p = 2$  and  $V = W_0^{m,2}(\Omega)$  provided that linearizations of the corresponding Euler-Lagrange equation (1.9) at any solution of it have no nontrivial solutions. Our new splitting lemmas in [39, 40] can be

effectively used to study periodic solutions of Lagrangian systems on compact manifolds which are strongly convex and has quadratic growth on the fibers, including the case of the system (1.4) if  $n = 1$  and Hypothesis  $\mathfrak{F}_{2,N}$  was satisfied. Their ideas were also used to derive the desired splitting and shifting lemmas for the Finsler energy functional on the space of  $H^1$ -curves in [41, 43]. However, when applying these splitting lemmas to the functional in (1.8) with  $p = 2$  we need that the involved critical points have higher smoothness, which can only be guaranteed under more assumptions on Lagrangian  $F$  by the regularity theory of differential equations. It is this unsatisfactory restriction that motivates us to look for a more suitable splitting lemma which is applicable to the functional in (1.3) under Hypothesis  $\mathfrak{F}_{p,N}$  with  $p = 2$ .

The following notation will be used throughout this paper. For normed linear spaces  $X, Y$  we denote by  $X^*$  the dual space of  $X$ , and by  $\mathcal{L}(X, Y)$  the space of linear bounded operators from  $X$  to  $Y$ . We also abbreviate  $\mathcal{L}(X) := \mathcal{L}(X, X)$ . The open ball in a normed linear space  $X$  with radius  $r$  and center in  $y \in X$  is denoted by  $B_X(y, r) := \{x \in X \mid \|x - y\|_X < r\}$  and the corresponding closed ball is written as  $\bar{B}_X(y, r) := \{x \in X \mid \|x - y\|_X \leq r\}$ . The (norm)-closure of a set  $S \subset X$  will be denoted by  $\bar{S}$  or  $Cl(S)$ . Let  $m$  and  $n$  be two positive integers,  $\Omega \subset \mathbb{R}^n$  a bounded domain with boundary  $\partial\Omega$ . Denote the general point of  $\Omega$  by  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and the element of Lebesgue  $n$ -measure on  $\Omega$  by  $dx$ . A *multi-index* is an  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N}_0)^n$ , where  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .  $|\alpha| := \alpha_1 + \dots + \alpha_n$  is called the length of  $\alpha$ . Denote by  $M(k)$  the number of such  $\alpha$  of length  $|\alpha| \leq k$ ,  $M_0(k) = M(k) - M(k-1)$ ,  $k = 0, \dots, m$ , where  $M(-1) = \emptyset$ . Then  $M(0) = M_0(0)$  only consists of  $\mathbf{0} = (0, \dots, 0) \in (\mathbb{N}_0)^n$ .

Let  $p \in [2, \infty)$  be a real number and  $N \geq 1$  an integer.

**Hypothesis  $\mathfrak{F}_{p,N}$ .** For each multi-index  $\gamma$  as above, let

$$p_\gamma \in (1, \infty) \text{ if } |\gamma| = m - n/p, \quad \text{and } p_\gamma = \frac{np}{n - (m - |\gamma|)p} \text{ if } m - n/p < |\gamma| \leq m,$$

$$q_\gamma = 1 \text{ if } |\gamma| < m - n/p, \quad \text{and } q_\gamma = \frac{p_\gamma}{p_\gamma - 1} \text{ if } m - n/p \leq |\gamma| \leq m;$$

and for each two multi-indexes  $\alpha, \beta$  as above, let  $p_{\alpha\beta} = p_{\beta\alpha}$  be defined by the conditions

$$p_{\alpha\beta} = 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta} \quad \text{if } |\alpha| = |\beta| = m,$$

$$p_{\alpha\beta} = 1 - \frac{1}{p_\alpha}, \quad \text{if } m - n/p \leq |\alpha| \leq m, \quad |\beta| < m - n/p,$$

$$p_{\alpha\beta} = 1 \quad \text{if } |\alpha|, |\beta| < m - n/p,$$

$$0 < p_{\alpha\beta} < 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta} \quad \text{if } |\alpha|, |\beta| \geq m - n/p, \quad |\alpha| + |\beta| < 2m.$$

For  $M_0(k) = M(k) - M(k-1)$ ,  $k = 0, 1, \dots, m$  as above, we write  $\xi \in \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)}$  as  $\xi = (\xi^0, \dots, \xi^m)$ , where  $\xi^0 = (\xi_0^1, \dots, \xi_0^N)^T \in \mathbb{R}^N$  and

$$\xi^k = (\xi_\alpha^i)_{\substack{1 \leq i \leq N \\ |\alpha| = k}} \in \mathbb{R}^{N \times M_0(k)} \quad \text{for } k = 1, \dots, m.$$

Denote by  $\xi_\alpha^k = \{\xi_\alpha^k : |\alpha| < m - n/p\}$  for  $k = 1, \dots, N$ . Suppose

$$\overline{\Omega} \times \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)} \ni (x, \xi) \mapsto F(x, \xi) \in \mathbb{R}$$

is a Caratheodory function (i.e., being measurable in  $x$  for all values of  $\xi$ , and continuous in  $\xi$  for almost all  $x$ ) with the following properties:

(i)  $F(x, \xi)$  is twice continuously differentiable in  $\xi$  for almost all  $x$ ,  $F(\cdot, 0) \in L^1(\Omega)$  and

$$F_\alpha^i(x, \xi) := \frac{\partial F(x, \xi)}{\partial \xi_\alpha^i}, \quad i = 1, \dots, N, \quad |\alpha| \leq m$$

satisfy:  $F_\alpha^i(\cdot, 0) \in L^1(\Omega)$  if  $|\alpha| < m - n/p$ , and  $F_\alpha^i(\cdot, 0) \in L^{q_\alpha}(\Omega)$  if  $m - n/p \leq |\alpha| \leq m$ ,  $i = 1, \dots, N$ .

(ii) There exists a continuous, positive, nondecreasing functions  $\mathbf{g}_1$  such that for  $i, j = 1, \dots, N$ ,  $|\alpha|, |\beta| \leq m$  and the above numbers  $p_{\alpha\beta}$  functions

$$\overline{\Omega} \times \mathbb{R}^{M(m)} \rightarrow \mathbb{R}, \quad (x, \xi) \mapsto F_{\alpha\beta}^{ij}(x, \xi) = \frac{\partial^2 F(x, \xi)}{\partial \xi_\alpha^i \partial \xi_\beta^j}$$

satisfy:

$$|F_{\alpha\beta}^{ij}(x, \xi)| \leq \mathbf{g}_1 \left( \sum_{k=1}^N |\xi_\alpha^k| \right) \left( 1 + \sum_{k=1}^N \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma^k|^{p_\gamma} \right)^{p_{\alpha\beta}}. \quad (1.1)$$

(iii) There exists a continuous, positive, nondecreasing functions  $\mathbf{g}_2$  such that

$$\sum_{|\alpha|=|\beta|=m} F_{\alpha\beta}^{ij}(x, \xi) \eta_\alpha^i \eta_\beta^j \geq \mathbf{g}_2 \left( \sum_{k=1}^N |\xi_\alpha^k| \right) \left( 1 + \sum_{k=1}^N \sum_{|\gamma|=m} |\xi_\gamma^k| \right)^{p-2} \sum_{i=1}^N \sum_{|\alpha|=m} (\eta_\alpha^i)^2 \quad (1.2)$$

for any  $\eta = (\eta_{\alpha\beta}^{ij}) \in \mathbb{R}^{N \times M_0(m)}$ .

*Note:* If  $m \leq n/p$  the functions  $\mathbf{g}_1$  and  $\mathbf{g}_2$  should be understand as positive constants.

For an element of  $W^{m,p}(\Omega, \mathbb{R}^N)$ ,  $\vec{u} = (u^1, \dots, u^N) : \Omega \rightarrow \mathbb{R}^N$ , we shall denote by  $D^k \vec{u}$  the set  $\{D^\alpha u^i : |\alpha| = k, i = 1, \dots, N\}$  for each  $k = 1, \dots, m$ , and form the expression  $F(x, \vec{u}(x), \dots, D^m \vec{u}(x))$ , in which  $\vec{u}(x)$  takes the place of  $\xi^0$ , and  $D^\alpha u^i(x)$  takes the place of  $\xi_\alpha^i$  respectively. Let  $V \subset W^{m,p}(\Omega, \mathbb{R}^N)$  be a closed subspace containing  $W_0^{m,p}(\Omega, \mathbb{R}^N)$ . Consider variational problem

$$\mathfrak{F}(\vec{u}) = \int_{\Omega} F(x, \vec{u}, \dots, D^m \vec{u}) dx, \quad \vec{u} \in V. \quad (1.3)$$

We call critical points of  $\mathfrak{F}$  *generalized solutions* of the boundary value problem corresponding to the subspace  $V$ :

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha F_\alpha^i(x, \vec{u}, \dots, D^m \vec{u}) = 0, \quad i = 1, \dots, N. \quad (1.4)$$

If  $N = 1$ , Hypothesis  $\mathfrak{F}_{p,N}$  can be written as the following simple version, which was first given in [58].

**Hypothesis  $\mathfrak{f}_p$ .** Let  $p, p_\alpha, q_\alpha, p_{\alpha\beta}$  and  $\Omega$  be as in Hypothesis  $\mathfrak{F}_{p,N}$ . Write  $\xi \in \mathbb{R}^{M(m)}$  as  $\xi = \{\xi_\alpha : |\alpha| \leq m\}$  and  $\xi_\circ = \{\xi_\alpha : |\alpha| < m - n/p\}$ . Suppose that  $f : \overline{\Omega} \times \mathbb{R}^{M(m)} \rightarrow \mathbb{R}$  is a Caratheodory function with the following properties:

(i)  $f(x, \xi)$  is twice continuously differentiable in  $\xi$  for almost all  $x$ ,  $f(\cdot, 0) \in L^1(\Omega)$  and each  $f_\alpha(x, \xi) := \frac{\partial f(x, \xi)}{\partial \xi_\alpha}$  satisfy:  $f_\alpha(\cdot, 0) \in L^1(\Omega)$  if  $|\alpha| < m - n/p$ , and  $f_\alpha(\cdot, 0) \in L^{q_\alpha}(\Omega)$  if  $m - n/p \leq |\alpha| \leq m$ .

(ii) There exists a continuous, positive, nondecreasing functions  $\mathfrak{g}_1$  such that for the above numbers  $p_{\alpha\beta}$  functions

$$\overline{\Omega} \times \mathbb{R}^{M(m)} \rightarrow \mathbb{R}, (x, \xi) \mapsto f_{\alpha\beta}(x, \xi) = \frac{\partial^2 f(x, \xi)}{\partial \xi_\alpha \partial \xi_\beta}$$

satisfy:

$$|f_{\alpha\beta}(x, \xi)| \leq \mathfrak{g}_1(|\xi_\circ|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}}. \quad (1.5)$$

(iii) There exists a continuous, positive, nondecreasing functions  $\mathfrak{g}_2$  such that

$$\sum_{|\alpha|=|\beta|=m} f_{\alpha\beta}(x, \xi) \eta_\alpha \eta_\beta \geq \mathfrak{g}_2(|\xi_\circ|) \left( 1 + \sum_{|\gamma|=m} |\xi_\gamma| \right)^{p-2} \sum_{|\alpha|=m} \eta_\alpha^2 \quad (1.6)$$

for any  $\eta \in \mathbb{R}^{M_0(m)}$ .

Consider the case  $m = 1$  and  $n \geq 2$ . Then  $M = n + 1$  and  $f$  becomes

$$f : \overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{R}, (x, \xi_0, \xi_1, \dots, \xi_n) \mapsto f(x, \xi_0, \xi_1, \dots, \xi_n).$$

The corresponding Hypothesis  $\mathfrak{f}_2$  is: there exist constant numbers  $c_1, c_2 > 0$  such that

$$|f_{ij}(x, \xi)| \leq c_1 \left( 1 + \sum_{k=0}^n |\xi_k|^{2_k} \right)^{2_{ij}}, \quad \sum_{i,j=1}^n f_{ij}(x, \xi) \eta_i \eta_j \geq c_2 \left( \sum_{i=1}^n \eta_i^2 \right). \quad (1.7)$$

Here, (I) if  $n = 2$ ,  $2_0 = s \in (2, \infty)$ ,  $2_i = 2$ ,  $i = 1, \dots, n$ ,  $2_{ij} = 0$  for  $i, j = 1, \dots, n$ ,  $2_{0i} = 2_{i0} \in (0, 1/2 - 1/s)$  for  $i = 1, \dots, n$ , and  $2_{00} \in (0, 1 - 2/s)$ ; (II) if  $n > 2$ ,  $2_0 = 2n/(n-2)$ ,  $2_i = 2$  for  $i = 1, \dots, n$ ,  $2_{ij} = 0$  for  $i, j = 1, \dots, n$ ,  $2_{i0} = 2_{0i} \in (0, 1/n)$  for  $i = 1, \dots, n$ ,  $2_{00} \in (0, 2/n)$ .

Under Hypothesis  $\mathfrak{f}_p$ , let  $V \subset W^{m,p}(\Omega)$  be any closed subspace containing  $W_0^{m,p}(\Omega)$ . The critical points of the variational problem

$$\mathcal{F}(u) = \int_{\Omega} f(x, u, \dots, D^m u) dx \quad (1.8)$$

in the Banach space  $V$  are called *generalized solutions* of the boundary value problem corresponding to the subspace  $V$ :

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, u, \dots, D^m u) = 0, \quad (1.9)$$

where  $D^k u(x) = \{D^\alpha u(x) : |\alpha| = k\}$ ,  $k = 1, \dots, m$ . For example, when  $V$  is  $W_0^{m,p}(\Omega)$  (resp.  $W^{m,p}(\Omega)$ ), the corresponding boundary value problem will be the Dirichlet (resp. Neumann) problem (cf. [60, pages 6-7]). Moreover, under Hypothesis  $\mathfrak{f}_2$ , if  $\dim \Omega = 2$  and  $f \in C^{k,\alpha}$  for some  $\alpha \in (0, 1)$  and an integer  $k \geq 3$ , it was proved in [60, Chapter 7, Th.4.4] that every critical point  $u$  of  $\mathcal{F}$  on  $W_0^{m,2}(\Omega)$  sits in  $C^{k+m-1,\alpha}(\overline{\Omega})$ ; in fact  $u$  is also analytic in  $\Omega$  provided that  $f$  is analytic in its arguments.

As stated on the pages 118-119 of [60] (see [58] for detailed arguments), under Hypothesis  $\mathfrak{f}_p$  the functional  $\mathcal{F}$  in (1.8) is of class  $C^1$ ; and the (derivative) mapping  $\mathcal{F}' : W_0^{m,p}(\Omega) \rightarrow [W_0^{m,p}(\Omega)]^*$  is Fréchet differentiable if  $p > 2$ , but only Gâteaux-differentiable if  $p = 2$ . A critical point  $u$  of  $\mathcal{F}$  is said to be *nondegenerate* if the derivative of  $\mathcal{F}'$  at it,  $\mathcal{F}''(u) : W_0^{m,p}(\Omega) \rightarrow \mathcal{L}(W_0^{m,p}(\Omega), [W_0^{m,p}(\Omega)]^*)$  is injective. In case  $p = 2$ , if  $\mathcal{F}$  has only nondegenerate critical points, Skrypnik [58, Chapter 5] obtained Morse inequalities provided that  $\mathcal{F}(u) \rightarrow +\infty$  as  $\|u\|_{m,2} \rightarrow \infty$ . On the other hand he also obtained

**Skrypnik Theorem** ([60, Chap.5, Sec. 5.1, Theorem 1]). *If  $p = 2$ ,  $m = 1$  and  $f \in C^2(\overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^n)$  has uniformly bounded mixed partial derivatives*

$$f_{ij} = \frac{\partial^2 f(x, u, \xi)}{\partial \xi_i \partial \xi_j}, \quad f_{i0} = \frac{\partial^2 f(x, u, \xi)}{\partial \xi_i \partial u}, \quad f_{00} = \frac{\partial^2 f(x, u, \xi)}{\partial u^2},$$

*(and therefore  $f$  satisfies Hypothesis  $\mathfrak{f}_2$ ), then the functional  $\mathcal{F}$  on  $W_0^{1,2}(\Omega)$  has Fréchet second derivative at zero if and only if*

$$f(x, 0, \xi) = \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j + \sum_{i=1}^n b_i(x) \xi_i + c(x).$$

So, generally speaking, under Hypothesis  $\mathfrak{f}_2$  the known Morse theory method cannot be effectively used to study critical points of  $\mathcal{F}$  on  $W_0^{m,2}(\Omega)$  without nondegenerate conditions. A similar question also appears in some optimal control problems [63].

The key of this paper is to prove a new splitting theorem (Theorem 2.2) for a class of non- $C^2$  functionals on a Hilbert space under the following Hypothesis 1.1 (following the notion and terminology in [40] without special statements). Even if for the Lagrangian systems studied in [39], we can largely simplify the arguments therein with this new theorem. However, the theories in [39, 40] may, sometime, provide more elaborate results as done in [44, 46, 47, 48].

**Hypothesis 1.1.** Let  $H$  be a Hilbert space with inner product  $(\cdot, \cdot)_H$  and the induced norm  $\|\cdot\|$ , and let  $X$  be a dense linear subspace in  $H$ . Let  $V$  be an open neighborhood of the origin  $\theta \in H$ ,

and let  $\mathcal{L} \in C^1(V, \mathbb{R})$  satisfy  $D\mathcal{L}(\theta) = 0$ . Assume that the gradient  $\nabla\mathcal{L}$  has a Gâteaux derivative  $B(u) \in \mathcal{L}_s(H)$  at every point  $u \in V \cap X$ , and that the map  $B : V \cap X \rightarrow \mathcal{L}_s(H)$  has a decomposition  $B = P + Q$ , where for each  $x \in V \cap X$ ,  $P(x) : H \rightarrow H$  is a bounded linear positive definite operator and  $Q(x) : H \rightarrow H$  is a compact linear operator with the following properties:

- (D1) All eigenfunctions of the operator  $B(\theta)$  that correspond to non-positive eigenvalues belong to  $X$ ;
- (D2) For any sequence  $\{x_k\}_{k \geq 1} \subset V \cap X$  with  $\|x_k\| \rightarrow 0$  it holds that  $\|P(x_k)u - P(\theta)u\| \rightarrow 0$  for any  $u \in H$ ;
- (D3) The map  $Q : V \cap X \rightarrow \mathcal{L}(H)$  is continuous at  $\theta$  with respect to the topology induced from  $H$  on  $V \cap X$ ;
- (D4) For any sequence  $\{x_n\}_{n \geq 1} \subset V \cap X$  with  $\|x_n\| \rightarrow 0$  (as  $n \rightarrow \infty$ ), there exist constants  $C_0 > 0$  and  $n_0 \in \mathbb{N}$  such that

$$(P(x_n)u, u)_H \geq C_0 \|u\|^2 \quad \forall u \in H, \quad \forall n \geq n_0.$$

(Note: In Lemma 2.9 we shall prove that the condition (D4) is equivalent to (D4\*) in [40]. Lemma 2.10 shows that this property with  $X = H$  is hereditary on closed subspaces).

Actually, we prove a more general parameterized splitting theorem, Theorem 2.18. Using it we complete generalizations of many bifurcation theorems for potential operators in Section 3. A weaker Marino-Prodi perturbation type result is also presented in Section 2.7. These constitute abstract theories in Part I of this paper. Part II deals with quasi-linear elliptic systems of higher order. In Section 4, we study fundamental analytic properties of the functional  $\mathfrak{F}$  under Hypothesis  $\mathfrak{F}_{p,N}$ . In particular, Hypothesis  $\mathfrak{F}_{2,N}$  assures that  $\mathfrak{F}$  satisfies Hypothesis 1.1 on any closed subspace of  $W^{m,2}(\Omega, \mathbb{R}^N)$  for a bounded Sobolev domain  $\Omega \subset \mathbb{R}^n$ . Because of these, the Morse theory methods can be used to study the quasi-linear elliptic boundary value problem (1.4) under Hypothesis  $\mathfrak{F}_{2,N}$  as done for the semi-linear elliptic one in [12, 52]. In other sections we are only satisfied to present some direct applications of results in Part I, for example, giving Morse inequalities in Section 6 and some bifurcation results for quasi-linear elliptic systems in Section 7. Further applications will be given in latter papers.

## Part I

# Abstract theories

## 2 Local Morse theory for a class of non- $C^2$ functionals

### 2.1 Statements of main results

Our local Morse theory mainly consist of a new splitting theorem and a Marino-Prodi perturbation type result for a class of non- $C^2$  functionals.

We always assume that Hypothesis 1.1 holds without special statements. Then it implies that  $\nabla \mathcal{L}$  is of class  $(S)_+$  near  $\theta$  as proved in [40, p.2966-2967]. In particular,  $\mathcal{L}$  satisfies the (PS) condition near  $\theta$ .

For the bounded linear self-adjoint operator  $B(\theta)$  on the Hilbert space  $H$ , let  $H = H^+ \oplus H^0 \oplus H^-$  be the orthogonal decomposition according to the positive definite, null and negative definite spaces of it. Denote by  $P^*$  the orthogonal projections onto  $H^*$ ,  $*$  = +, 0, -. By [40, Proposition B.2] the above fundamental assumptions implies that there exists a constant  $C_0 > 0$  such that each  $\lambda \in (-\infty, C_0)$  is either not in the spectrum  $\sigma(B(\theta))$  or is an isolated point of  $\sigma(B(\theta))$  which is also an eigenvalue of finite multiplicity. It follows that both  $H^0$  and  $H^-$  are finitely dimensional, and that there exists a small  $a_0 > 0$  such that  $[-2a_0, 2a_0] \cap \sigma(B(\theta))$  at most contains a point 0, and hence

$$\left. \begin{aligned} (B(\theta)u, u)_H &\geq 2a_0\|u\|^2 \quad \forall u \in H^+, \\ (B(\theta)u, u)_H &\leq -2a_0\|u\|^2 \quad \forall u \in H^-. \end{aligned} \right\} \quad (2.1)$$

Note that (D1) implies  $H^- \oplus H^0 \subset X$ .  $\nu := \dim H^0$  and  $\mu := \dim H^-$  are called the *Morse index* and *nullity* of the critical point  $\theta$ . In particular, if  $\nu = 0$  the critical point  $\theta$  is said to be *nondegenerate*. Without special statements, all nondegenerate critical points in this paper are in the sense of this definition. Moreover, such a critical point must be isolated by (2.4) or (2.5)

Our first result is the following Morse-Palais Lemma. Comparing with that of [40, Remark 2.2(i)], the smoothness of  $\mathcal{L}$  is strengthened, but other conditions are suitably weakened.

**Theorem 2.1.** *Under Hypothesis 1.1, if  $\nu = 0$ , i.e.,  $\theta$  is nondegenerate, there exist a small  $\epsilon > 0$ , an open neighborhood  $W$  of  $\theta$  in  $H$  and an origin-preserving homeomorphism,  $\phi : B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon) \rightarrow W$ , such that*

$$\mathcal{L} \circ \phi(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2, \quad \forall (u^+, u^-) \in B_{H^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon).$$

Moreover, if  $\hat{H}$  is a closed subspace containing  $H^-$ , and  $\hat{H}^+$  is the orthogonal complement of  $H^-$  in  $\hat{H}$ , i.e.,  $\hat{H}^+ = \hat{H} \cap H^+$ , then  $\phi$  restricts to a homeomorphism  $\hat{\phi} : (B_{\hat{H}^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon)) \rightarrow \hat{W} := W \cap \hat{H}$ , and  $\mathcal{L} \circ \hat{\phi}(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2$  for all  $(u^+, u^-) \in B_{\hat{H}^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon)$ .

Under the assumptions of this theorem, if  $X = H$  we can prove that  $\nabla \mathcal{L}$  is locally invertible at  $\theta$  in Theorem 2.15. Theorem 2.1 is also key for us to prove the following degenerate case.



**Theorem 2.2** (Splitting Theorem). *Let Hypothesis 1.1 hold with  $X = H$ . Suppose  $\nu \neq 0$ . Then there exist small positive numbers  $\epsilon, r, s$ , a unique continuous map  $\varphi : B_{H^0}(\theta, \epsilon) \rightarrow H^+ \oplus H^-$  satisfying  $\varphi(\theta) = \theta$  and*

$$(I - P^0)\nabla\mathcal{L}(z + \varphi(z)) = 0 \quad \forall z \in B_{H^0}(\theta, \epsilon), \quad (2.2)$$

*an open neighborhood  $W$  of  $\theta$  in  $H$  and an origin-preserving homeomorphism*

$$\Phi : B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, r) + B_{H^-}(\theta, s)) \rightarrow W$$

*of form  $\Phi(z, u^+ + u^-) = z + \varphi(z) + \phi_z(u^+ + u^-)$  with  $\phi_z(u^+ + u^-) \in H^+ \oplus H^-$  such that*

$$\mathcal{L} \circ \Phi(z, u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 + \mathcal{L}(z + \varphi(z))$$

*for all  $(z, u^+ + u^-) \in B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, r) + B_{H^-}(\theta, s))$ . Moreover,  $\varphi$  is of class  $C^{1-0}$ , and the homeomorphism  $\Phi$  has also properties:*

- (a) *For each  $z \in B_{H^0}(\theta, \epsilon)$ ,  $\Phi(z, \theta) = z + \varphi(z)$ ,  $\phi_z(u^+ + u^-) \in H^-$  if and only if  $u^+ = \theta$ ;*
- (b) *The functional  $B_{H^0}(\theta, \epsilon) \ni z \mapsto \mathcal{L}^\circ(z) := \mathcal{L}(z + \varphi(z))$  is of class  $C^1$  and*

$$D\mathcal{L}^\circ(z)v = D\mathcal{L}(z + \varphi(z))v, \quad \forall v \in H^0.$$

*If  $\mathcal{L}$  is of class  $C^{2-0}$ , so is  $\mathcal{L}^\circ$ .*

Since the map  $\varphi$  satisfying (2.2) is unique, as [39, 40] it is possible to prove in some cases that  $\varphi$  and  $\mathcal{L}^\circ$  are of class  $C^1$  and  $C^2$ , respectively.

Theorem 2.2 is a direct consequence of Theorems 2.14, 2.18 and Proposition 2.17.

Under the assumptions of Theorem 2.2, we cannot assure that  $\theta$  is an isolated critical point. But, if  $x \in H$  is a critical point of  $\mathcal{L}$  and very close to  $\theta$ , it follows from (2.2) and (2.10)–(2.11) that  $x \in H^0$  and satisfies  $\varphi(x) = \theta$ .

Theorems 2.1, 2.2 cannot be derived from those of [24]. In fact, according to the conditions (c) and (d) in [24, Theorem 1.3] the functional  $\mathcal{L}$  in Theorem 2.1 should satisfy:

- (c')  $\exists \eta > 0, \delta > 0$  such that  $|(B(u)(u + z) - B(\theta)(u + z), h)| < \eta\|u + z\| \cdot \|h\|$  for all  $u \in B_H(\theta, \delta), z \in H^0$  and  $h \in H \setminus \{\theta\}$ ;
- (d')  $\exists \delta > 0$  such that  $(\nabla\mathcal{L}(z + u_1^+ + u_1^-) - \nabla\mathcal{L}(z + u_2^+ + u_2^-), (u_1^+ - u_2^+) + (u_1^- - u_2^-)) > 0$  for all  $(u_1^+, u_1^-), (u_2^+, u_2^-) \in B_{H^+}(\theta, \delta) \times B_{H^-}(\theta, \delta)$  with  $u_1^+ + u_1^- \neq u_2^+ + u_2^-$ .

The former implies  $\|B(u)(u + z) - B(\theta)(u + z)\| \leq \eta\|u + z\|$  for all  $u \in B_H(\theta, \delta), z \in H^0$ ; and specially

$$\begin{aligned} \|B(u)u - B(\theta)u\| &\leq \eta\|u\| \quad \forall u \in B_H(\theta, \delta), \\ \|B(z)z - B(\theta)z\| &\leq \eta\|z\| \quad \forall z \in B_H(\theta, \delta) \cap H^0. \end{aligned}$$

The latter implies that for some  $t = t(z, u_1^+, u_1^-, u_2^+, u_2^-) \in (0, 1)$ ,

$$(B(z + u_2^+ + u_2^- + tu^+ + tu^-)(u^+ + u^-), u^+ + u^-) > 0$$

with  $u^+ = u_1^+ - u_2^+$  and  $u^- = u_1^- - u_2^-$ .

From these it is not hard to see that under our assumptions the conditions (c') and (d') cannot be satisfied in general.

Let  $H_q(A, B; \mathbf{K})$  denote the  $q$ th relative singular homology group of a pair  $(A, B)$  of topological spaces with coefficients in Abel group  $\mathbf{K}$ . For each  $q \in \mathbb{N} \cup \{0\}$  the  $q$ th critical group (with coefficients in  $\mathbf{K}$ ) of  $\mathcal{L}$  at a point  $\theta$  is defined by

$$C_q(\mathcal{L}, \theta; \mathbf{K}) = H_q(\mathcal{L}_c \cap U, \mathcal{L}_c \cap U \setminus \{\theta\}; \mathbf{K}),$$

where  $c = \mathcal{L}(\theta)$ ,  $\mathcal{L}_c = \{\mathcal{L} \leq c\}$  and  $U$  is a neighborhood of  $\theta$  in  $H$ .

Under the assumptions of Theorem 2.1 we have  $C_q(\mathcal{L}, \theta; \mathbf{K}) = \delta_{q\mu} \mathbf{K}$  as usual. For the degenerate case, though our  $\mathcal{L}^\circ$  is only of class  $C^1$ , the proofs in [50, Theorem 8.4] and [14, Theorem 5.1.17] (or [13, Theorem I.5.4]) may be slightly modify to get the following shifting theorem, a special case of Theorem 2.19.

**Theorem 2.3** (Shifting Theorem). *Under the assumptions of Theorem 2.2, if  $\theta$  is an isolated critical point of  $\mathcal{L}$ , for any Abel group  $\mathbf{K}$  it holds that*

$$C_q(\mathcal{L}, \theta; \mathbf{K}) \cong C_{q-\mu}(\mathcal{L}^\circ, \theta; \mathbf{K}) \quad \forall q = 0, 1, \dots,$$

As done for  $C^2$  functionals in [13, 14, 50, 52] some critical point theorems can be derived from Theorem 2.3. For example,  $C_q(\mathcal{L}, \theta; \mathbf{K})$  is equal to  $\delta_{q\mu} \mathbf{K}$  (resp.  $\delta_{q(\mu+\nu)} \mathbf{K}$ ) if  $\theta$  is a local minimizer (resp. maximizer) of  $\mathcal{L}^\circ$ , and  $C_q(\mathcal{L}, \theta; \mathbf{K}) = 0$  for  $q \leq \mu$  and  $q \geq \mu + \nu$  if  $\theta$  is neither a local minimizer nor local maximizer of  $\mathcal{L}^\circ$ . Similarly, the corresponding generalizations of Theorems 2.1, 2.1', 2.2, 2.3 and Corollary 1.3 in [13, Chapter II] can be obtained with Theorems 2.1, 2.2 and their equivariant versions in Section 2.6. In particular, as a generalization of [13, Theorem II.1.6] (or [14, Theorem 5.1.20]) we have

**Theorem 2.4.** *Let Hypothesis 1.1 hold with  $X = H$ , and let  $\theta$  be an isolated critical point of mountain pass type, i.e.,  $C_1(\mathcal{L}, \theta; \mathbf{K}) \neq 0$ . Suppose that  $\nu > 0$  and  $\mu = 0$  imply  $\nu = 1$ . Then  $C_q(\mathcal{L}, \theta; \mathbf{K}) = \delta_{q1} \mathbf{K}$ .*

When  $\nu > 0$  and  $\mu = 1$ ,  $C_0(\mathcal{L}^\circ, \theta; \mathbf{K}) \neq 0$  by Theorem 2.3. We can change  $\mathcal{L}^\circ$  outside a very small neighborhood  $\theta \in B_{H^0}(\theta, \epsilon)$  to get a  $C^1$  functional on  $H^0$  which is coercive (and so satisfies the (PS)-condition). Then it follows from  $C_0(\mathcal{L}^\circ, \theta; \mathbf{K}) \neq 0$  and [52, Proposition 6.95] that  $\theta$  is a local minimizer of  $\mathcal{L}^\circ$ .

As a generalization of Corollary 3.1 in [13, page 102] we have also: Under the assumptions of Theorem 2.4, if the smallest eigenvalue  $\lambda_1$  of  $B(\theta) = d^2\mathcal{L}(\theta)$  is simple whenever  $\lambda_1 = 0$ , then  $\lambda_1 \leq 0$ , and  $\text{index}(\nabla \mathcal{L}, \theta) = -1$ .

Theorem 5.1 and Corollary 5.1 in [13, page 121] are also true if “ $f \in C^2(M, \mathbb{R})$ ” and “Fredholm operators  $d^2f(x_i)$ ” are replaced by “ $f \in C^1(M, \mathbb{R})$  and  $\nabla f$  is Gâteaux differentiable” and “under some chart around  $p_i$  the functional  $f$  has a representation that satisfies Hypothesis 1.1”, respectively.

Marino and Prodi [49] studied local Morse function approximations for  $C^2$  functionals on Hilbert spaces. We shall generalize their result to a class of functionals satisfying the following stronger assumption than Hypothesis 1.1.

**Hypothesis 2.5.** Let  $V$  be an open set of a Hilbert space  $H$  with inner product  $(\cdot, \cdot)_H$ , and  $\mathcal{L} \in C^1(V, \mathbb{R})$ . Assume that the gradient  $\nabla \mathcal{L}$  has a Gâteaux derivative  $B(u) \in \mathcal{L}_s(H)$  at every point  $u \in V$ , and that the map  $B : V \rightarrow \mathcal{L}_s(H)$  has a decomposition  $B = P + Q$ , where for each  $u \in V$ ,  $P(u) : H \rightarrow H$  is a bounded linear positive definitive operator and  $Q(u) : H \rightarrow H$  is a compact linear operator with the following properties:

- (i) For any  $u \in H$ , the map  $V \ni x \mapsto P(x)u \in H$  is continuous;
- (ii) The map  $Q : V \rightarrow \mathcal{L}(H)$  is continuous;
- (iii)  $P$  is local positive definite uniformly, i.e., each  $x_0 \in V$  has a neighborhood  $\mathcal{U}(x_0)$  such that for some constants  $C_0 > 0$ ,

$$(P(x)u, u)_H \geq C_0 \|u\|^2, \quad \forall u \in H, \forall x \in \mathcal{U}(x_0).$$

As in the proof of Theorem 4.1, under Hypothesis  $\mathfrak{F}_{2,N}$ , we can check that the functional  $\mathfrak{F}$  in (1.3) satisfies this hypothesis. By improving methods in [49, 13, 20] we may prove

**Theorem 2.6.** *Under Hypothesis 2.5, suppose: (a)  $u_0 \in V$  is a unique critical point of  $\mathcal{L}$ , (b) the corresponding maps  $\varphi$  and  $\mathcal{L}^\circ$  as in Theorem 2.2 near  $u_0$  are of classes  $C^1$  and  $C^2$ , respectively, (c)  $\mathcal{L}$  satisfies the (PS) condition. Then for any  $\epsilon > 0$  and  $r > 0$  such that  $\bar{B}_H(u_0, r) \subset V$  and there exists a functional  $\tilde{\mathcal{L}} \in C^1(V, \mathbb{R})$  with the following properties:*

- (i)  $\tilde{\mathcal{L}}$  satisfies Hypothesis 2.5 and the (PS) condition;
- (ii)  $\sup_{u \in V} \|\mathcal{L}^{(i)}(u) - \tilde{\mathcal{L}}^{(i)}(u)\| < \epsilon$ ,  $i = 0, 1, 2$ ;
- (iii)  $\mathcal{L}(x) = \tilde{\mathcal{L}}(x)$  if  $x \in V$  and  $\|x - u_0\| \geq r$ ;
- (iv) the critical points of  $g$ , if any, are in  $B_H(u_0, r)$  and nondegenerate (so finitely many by the arguments below 2.1); moreover the Morse indexes of these critical points sit in  $[m^-, m^- + n^0]$ , where  $m^-$  and  $n^0$  are the Morse index and nullity of  $u_0$ , respectively.

As showed, the functionals in [39, 48] satisfy the conditions of this theorem. Marino–Prodi’s result has many important applications in the critical point theory, see [13, 20, 30, 38] and literature therein. With Theorem 2.6 they may be given in our framework.

Marino–Prodi’s perturbation theorem in [49] was also generalized to the equivariant case under the finite (resp. compact Lie) group action by Wasserman [69] (resp. Viterbo [66]), see the proof of Theorem 7.8 in [13, Chapter I] for full details. Similarly, we can present an equivariant version of Theorem 2.6 for compact Lie group action, but it is omitted here.

**Strategies of proofs for results in this section and arrangements.** Under the assumptions of Theorem 2.1, no known implicit function theorems or contraction mapping principles can be used to get  $\varphi$  in (2.2), which is different from the case in [39, 40]. The methods in [24] provide a possible way to construct such a  $\varphi$ . However, as mentioned above our assumptions cannot guarantee the above conditions (c') and (d'). Fortunately, it is with Lemma 2.12 and Theorem 2.1 that we can complete this construction.

In Section 2.2 we list some lemmas. Theorem 2.1 will be proved in Section 2.3. It is necessary for a key implicit function theorem for a family of potential operators, Theorem 2.14, which is proved in Section 2.4; we also give an inverse function theorem, Theorem 2.15, there. In Section 2.5 we shall prove a parameterized splitting theorem, Theorem 2.18, and a parameterized shifting theorem, Theorem 2.19; Theorems 2.2, 2.3 are special cases of them, respectively. The equivariant case is considered in Section 2.6. Theorem 2.6 will be proved in Section 2.7.

## 2.2 Lemmas

Under Hypothesis 1.1 we have the following two lemmas as proved in [39, 40].

**Lemma 2.7.** *There exists a function  $\omega : V \cap X \rightarrow [0, \infty)$  such that  $\omega(x) \rightarrow 0$  as  $x \in V \cap X$  and  $\|x\| \rightarrow 0$ , and that*

$$|(B(x)u, v)_H - (B(\theta)u, v)_H| \leq \omega(x)\|u\| \cdot \|v\|$$

for any  $x \in V \cap X$ ,  $u \in H^0 \oplus H^-$  and  $v \in H$ .

**Lemma 2.8.** *There exists a small neighborhood  $U \subset V$  of  $\theta$  in  $H$  and a number  $a_1 \in (0, 2a_0]$  such that for any  $x \in U \cap X$ ,*

- (i)  $(B(x)u, u)_H \geq a_1\|u\|^2 \forall u \in H^+;$
- (ii)  $|(B(x)u, v)_H| \leq \omega(x)\|u\| \cdot \|v\| \forall u \in H^+, \forall v \in H^- \oplus H^0;$
- (iii)  $(B(x)u, u)_H \leq -a_0\|u\|^2 \forall u \in H^-.$

**Lemma 2.9.** *Actually, (D4) is equivalent to the condition (D\*) in [40], i.e.,*

**(D4\*)** *There exist positive constants  $\eta_0 > 0$  and  $C'_0 > 0$  such that*

$$(P(x)u, u) \geq C'_0\|u\|^2 \quad \forall u \in H, \forall x \in B_H(\theta, \eta_0) \cap X.$$

*Proof.* Indeed, since each  $P(x)$  is a positive definite linear operator, its spectral set is a bounded closed subset in  $(0, \infty)$ . Moreover, we have  $\sigma(\sqrt{P(x)}) = \{\sqrt{\lambda} \mid \lambda \in \sigma(P(x))\}$ . So (D4) is equivalent to the statement: For any sequence  $\{x_n\} \subset V \cap X$  with  $\|x_n\| \rightarrow 0$  (as  $n \rightarrow \infty$ ), there holds:  $\inf_n \min \sigma(\sqrt{P(x_n)}) > 0$ . Similarly, (D4\*) can be equivalently expressed as: There exist positive constants  $\eta_0 > 0$  such that

$$\inf\{\min \sigma(\sqrt{P(x)}) \mid x \in B_H(\theta, \eta_0) \cap X\} > 0.$$

Suppose (D4) holds. Since  $\eta \mapsto \inf\{\min \sigma(\sqrt{P(x)}) \mid x \in B_H(\theta, \eta) \cap X\}$  is non-increasing, that (D4) does not hold means that there exists a sequence  $\{x_n\} \subset V \cap X$  with  $\|x_n\| \rightarrow 0$  (as  $n \rightarrow \infty$ ) such that  $\inf_n \min \sigma(\sqrt{P(x_n)}) \rightarrow 0$ , which contradicts (D4).  $\square$

**Lemma 2.10.** *Suppose that Hypothesis 1.1 with  $X = H$  is satisfied. Then for any closed subspace  $\hat{H} \subset H$ ,  $(\hat{H}, \hat{V}, \hat{\mathcal{L}})$  satisfies Hypothesis 1.1 with  $X = H$ , where  $\hat{V} := V \cap \hat{H}$  and  $\hat{\mathcal{L}} := \mathcal{L}|_{\hat{V}}$ .*

*Proof.* Clearly,  $\hat{\mathcal{L}} \in C^1(\hat{V}, \mathbb{R})$  and  $D\hat{\mathcal{L}}(\theta) = 0$ . Denote by  $\Pi : H \rightarrow \hat{H}$  the orthogonal projection. Then the gradient of  $\hat{\mathcal{L}}$  at  $u \in \hat{V}$ ,  $\nabla \hat{\mathcal{L}}(u)$ , is equal to  $\Pi \nabla \mathcal{L}(u)$ . It follows that  $\nabla \hat{\mathcal{L}}$  at any  $u \in \hat{V}$  has a Gâteaux derivative  $\hat{B}(u) = \Pi \circ B(u)|_{\hat{H}} \in \mathcal{L}_s(\hat{H})$ . For any  $u \in \hat{V}$ , put  $\hat{P}(u) = \Pi \circ P(u)|_{\hat{H}}$  and  $\hat{Q}(u) = \Pi \circ Q(u)|_{\hat{H}}$ . Then  $\hat{B} = \hat{P} + \hat{Q} : \hat{V} \rightarrow \mathcal{L}_s(\hat{H})$ ,  $\hat{P}(u)$  is positive definite, and  $\hat{Q}(u)$  is a compact linear operator. It is easily checked that other conditions are satisfied.  $\square$

### 2.3 Proof of Theorem 2.1

Take a small  $\epsilon > 0$  so that  $\bar{B}_{H^+}(\theta, \epsilon) \oplus \bar{B}_{H^-}(\theta, \epsilon)$  is contained in the open neighborhood  $U$  in Lemma 2.8. Let us prove the  $C^1$  functional

$$\bar{B}_{H^+}(\theta, \epsilon) \oplus \bar{B}_{H^-}(\theta, \epsilon) \rightarrow \mathbb{R}, \quad u^+ + u^- \mapsto \mathcal{L}(u^+ + u^-)$$

satisfies the conditions in [23, Theorem 1.1].

**Step 1.** Fix  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon) \cap X$  and  $u_1^-, u_2^- \in \bar{B}_{H^-}(\theta, \epsilon)$  (which are contained in  $X$  by (D1)). Since  $\nabla \mathcal{L}$  have a Gâteaux derivative  $B(u) \in \mathcal{L}_s(H)$  at every point  $u \in V \cap X$ , the function

$$V \rightarrow \mathbb{R}, \quad u \mapsto (\nabla \mathcal{L}(u^+ + u), u_2^- - u_1^-)_H$$

is Gâteaux differentiable at every  $u \in V \cap X$ . Using the mean value theorem we have  $t \in (0, 1)$  such that

$$\begin{aligned} & (\nabla \mathcal{L}(u^+ + u_2^-), u_2^- - u_1^-)_H - (\nabla \mathcal{L}(u^+ + u_1^-), u_2^- - u_1^-)_H \\ &= (B(u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-)_H \\ &\leq -a_0 \|u_2^- - u_1^-\|^2 \end{aligned}$$

by Lemma 2.8(iii). Note that  $\bar{B}_{H^+}(\theta, \epsilon) \cap X$  is dense in  $\bar{B}_{H^+}(\theta, \epsilon)$  and  $\nabla \mathcal{L}$  is continuous. For all  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon)$  and  $u_i^- \in \bar{B}_{H^-}(\theta, \epsilon)$ ,  $i = 1, 2$ , we deduce

$$(\nabla \mathcal{L}(u^+ + u_2^-), u_2^- - u_1^-)_H - (\nabla \mathcal{L}(u^+ + u_1^-), u_2^- - u_1^-)_H \leq -a_0 \|u_2^- - u_1^-\|^2. \quad (2.3)$$

This implies the condition (ii) of [23, Theorem 1.1].

**Step 2.** Let  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon) \cap X$  and  $u^- \in \bar{B}_{H^-}(\theta, \epsilon)$  (which is contained in  $X$  by (D1)). Then

since  $D\mathcal{L}(\theta) = 0$ , by the mean value theorem, for some  $t \in (0, 1)$  we have

$$\begin{aligned}
& D\mathcal{L}(u^+ + u^-)(u^+ - u^-) \\
&= (\nabla\mathcal{L}(u^+ + u^-), u^+ - u^-)_H - (\nabla\mathcal{L}(\theta), u^+ - u^-)_H \\
&= (B(t(u^+ + u^-))(u^+ + u^-), u^+ - u^-)_H \\
&= (B(t(u^+ + u^-))u^+, u^+)_H - (B(t(u^+ + u^-))u^-, u^-)_H \\
&\geq a_1\|u^+\|^2 + a_0\|u^-\|^2
\end{aligned} \tag{2.4}$$

by Lemma 2.8(i) and (iii). As above (2.4) also holds for all  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon)$  because  $\bar{B}_H(\theta, \epsilon) \cap X^+$  is dense in  $\bar{B}_H(\theta, \epsilon) \cap H^+$ . Hence  $D\mathcal{L}(u^+ + u^-)(u^+ - u^-) > 0$  for  $(u^+, u^-) \neq (\theta, \theta)$ . (This implies  $\theta$  to be an isolated critical point of  $\mathcal{L}$ ). The condition (iii) of [23, Theorem 1.1] is satisfied.

**Step 3.** For  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon) \cap X$ , as above we have  $t \in (0, 1)$  such that

$$\begin{aligned}
D\mathcal{L}(u^+)u^+ &= D\mathcal{L}(u^+)u^+ - D\mathcal{L}(\theta)u^+ \\
&= (\nabla\mathcal{L}(u^+), u^+)_H - (\nabla\mathcal{L}(\theta), u^+)_H \\
&= (B(tu^+)u^+, u^+)_H \\
&\geq a_1\|u^+\|^2
\end{aligned} \tag{2.5}$$

because of Lemma 2.8(i). It follows that

$$D\mathcal{L}(u^+)u^+ \geq a_1\|u^+\|^2 > p(\|u^+\|) \quad \forall u^+ \in \bar{B}_{H^+}(\theta, \epsilon) \setminus \{\theta\},$$

where  $p : (0, \epsilon] \rightarrow (0, \infty)$  is a non-decreasing function given by  $p(t) = \frac{a_1}{2}t^2$ . Hence the condition (iv) of [23, Theorem 1.1] is satisfied.

For the second claim, note that (2.3)–(2.5) hold for all  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon)$  and  $u^-, u_i^- \in \bar{B}_{H^+}(\theta, \epsilon)$ ,  $i = 1, 2$ . Of course, they are still true for all  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon)$ . Carefully checking the proof of [23, Theorem 1.1] the conclusion is easily obtained. (Note that this claim seems unable to be directly derived from Lemma 2.10.)  $\square$

Actually, from the proof of Theorem 2.1 we may get the more general claim, which is needed for later applications.

**Theorem 2.11.** *Under Hypothesis 1.1, let  $\mathcal{G} \in C^1(V, \mathbb{R})$  satisfy: i)  $\mathcal{G}'(\theta) = \theta$ , ii) the gradient  $\nabla\mathcal{G}$  has Gâteaux derivative  $\mathcal{G}''(u) \in \mathcal{L}_s(H)$  at any  $u \in V$ , and  $\mathcal{G}'' : V \rightarrow \mathcal{L}_s(H)$  are continuous at  $\theta$ . Suppose  $\text{Ker}(B(\theta)) = \{\theta\}$ , i.e.,  $\theta$  is a nondegenerate critical point of  $\mathcal{L}$ . Then there exist  $\rho > 0$ ,  $\epsilon > 0$ , a family of open neighborhoods of  $\theta$  in  $H$ ,  $\{W_\lambda \mid |\lambda| \leq \rho\}$  and a family of origin-preserving homeomorphisms,  $\phi_\lambda : B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon) \rightarrow W_\lambda$ ,  $|\lambda| \leq \rho$ , such that*

$$(\mathcal{L} + \lambda\mathcal{G}) \circ \phi_\lambda(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2$$

for all  $(u^+, u^-) \in B_{H^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon)$ . Moreover,  $[-\rho, \rho] \times (B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon)) \ni (\lambda, u) \mapsto \phi_\lambda(u) \in H$  is continuous.

*Proof.* Since  $\mathcal{G}'' : V \rightarrow \mathcal{L}_s(H)$  are continuous at  $\theta$ , as in the proof of (2.3) we may shrink  $\epsilon > 0$  and find  $\rho > 0$  such that for all  $\lambda \in [-\rho, \rho]$ ,  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon)$  and  $u_i^- \in \bar{B}_{H^-}(\theta, \epsilon)$ ,  $i = 1, 2$ ,

$$|\lambda(\nabla \mathcal{G}(u^+ + u_2^-), u_2^- - u_1^-)_H - \lambda(\nabla \mathcal{G}(u^+ + u_1^-), u_2^- - u_1^-)_H| \leq \frac{a_0}{2} \|u_2^- - u_1^-\|^2.$$

This and (2.3) lead to

$$\begin{aligned} & (\nabla(\mathcal{L} + \lambda \mathcal{G})(u^+ + u_2^-), u_2^- - u_1^-)_H - (\nabla(\mathcal{L} + \lambda \mathcal{G})(u^+ + u_1^-), u_2^- - u_1^-)_H \\ & \leq -\frac{a_0}{2} \|u_2^- - u_1^-\|^2. \end{aligned} \quad (2.6)$$

Similarly, as in the proof of (2.4) we may shrink the above  $\rho > 0$  and  $\epsilon > 0$  so that

$$|\lambda D\mathcal{G}(u^+ + u^-)(u^+ - u^-)| \leq \frac{a_1}{2} \|u^+\|^2 + \frac{a_0}{2} \|u^-\|^2$$

for all  $\lambda \in [-\rho, \rho]$ ,  $u^+ \in \bar{B}_{H^+}(\theta, \epsilon)$  and  $u^- \in \bar{B}_{H^-}(\theta, \epsilon)$ . This and (2.4) yield

$$D(\mathcal{L} + \lambda \mathcal{G})(u^+ + u^-)(u^+ - u^-) \geq \frac{a_1}{2} \|u^+\|^2 + \frac{a_0}{2} \|u^-\|^2$$

and specially  $D(\mathcal{L} + \lambda \mathcal{G})(u^+)(u^+) \geq \frac{a_1}{2} \|u^+\|^2$ . These and (2.6) show that the conditions of [40, Theorem A.1] are satisfied. The desired conclusions follow immediately.  $\square$

## 2.4 An implicit function theorem for a family of potential operators

In this subsection we shall prove an implicit function theorem, Theorem 2.14, which implies the first claim in Theorem 2.2. We also give an inverse function theorem, Theorem 2.15, though it is not used in this paper.

Without special statements, we always assume that Hypothesis 1.1 holds in this subsection.

Take  $\epsilon > 0$ ,  $r > 0$  and  $s > 0$  so small that the closures of both

$$\mathcal{Q}_{r,s} := B_{H^+}(\theta, r) \oplus B_{H^-}(\theta, s) \quad \text{and} \quad B_{H^0}(\theta, \epsilon) \oplus \mathcal{Q}_{r,s}$$

are contained in the neighborhood  $U$  in Lemma 2.8. Since  $H^0 \subset X$ ,  $X \cap \mathcal{Q}_{r,s}$  is also dense in  $\mathcal{Q}_{r,s}$ . Let  $P^\perp = I - P^0 = P^+ + P^-$ . By Lemma 2.8 we obtain  $a'_0 > 0$ ,  $a'_1 > 0$  such that

$$(P^\perp \nabla \mathcal{L}(z + u), u^+)_H = (\nabla \mathcal{L}(u), u^+)_H \geq a'_1 \|u^+\|^2 - a'_0 [\omega(z + u)]^2 \|u^-\|^2, \quad (2.7)$$

$$(P^\perp \nabla \mathcal{L}(z + u), u^-)_H = (\nabla \mathcal{L}(u), u^-)_H \leq -a'_1 \|u^-\|^2 + a'_0 [\omega(z + u)]^2 \|u^+\|^2 \quad (2.8)$$

for all  $u \in \overline{\mathcal{Q}_{r,s}}$  and  $z \in \bar{B}_{H^0}(\theta, \epsilon)$ . Since  $\omega(z + u) \rightarrow 0$  as  $\|z + u\| \rightarrow 0$ , by shrinking  $r > 0$ ,  $s > 0$  and  $\epsilon > 0$  we can require

$$[\omega(z + u)]^2 < \frac{a'_1}{2a'_0}, \quad \forall (z, u) \in \bar{B}_{H^0}(\theta, \epsilon) \times \overline{\mathcal{Q}_{r,s}}. \quad (2.9)$$

Then this and (2.7)–(2.8) lead to

$$(P^\perp \nabla \mathcal{L}(z + u), u^+)_H \geq a'_1 \|u^+\|^2 - \frac{a'_1}{2} \|u^-\|^2, \quad (2.10)$$

$$(P^\perp \nabla \mathcal{L}(z + u), u^-)_H \leq -a'_1 \|u^-\|^2 + \frac{a'_1}{2} \|u^+\|^2 \quad (2.11)$$

for all  $u \in \overline{\mathcal{Q}_{r,s}}$  and  $z \in \bar{B}_{H^0}(\theta, \epsilon)$ , and hence

$$(tP^\perp \nabla \mathcal{L}(z_1 + u) + (1-t)P^\perp \nabla \mathcal{L}(z_2 + u), u^+)_H \geq a'_1 \|u^+\|^2 - \frac{a'_1}{2} \|u^-\|^2, \quad (2.12)$$

$$(tP^\perp \nabla \mathcal{L}(z_1 + u) + (1-t)P^\perp \nabla \mathcal{L}(z_2 + u), u^-)_H \leq -a'_1 \|u^-\|^2 + \frac{a'_1}{2} \|u^+\|^2 \quad (2.13)$$

for all  $u \in \overline{\mathcal{Q}_{r,s}}$  and  $z_j \in \bar{B}_{H^0}(\theta, \epsilon)$ ,  $j = 1, 2$ , and  $t \in [0, 1]$ .

**Lemma 2.12.** *If  $r > 0, s > 0$  and  $\epsilon > 0$  are so small that (2.9) is satisfied, then*

$$\inf\{\|tP^\perp \nabla \mathcal{L}(z_1 + u) + (1-t)P^\perp \nabla \mathcal{L}(z_2 + u)\| \mid (t, z_1, z_2, u) \in \Omega\} > 0,$$

where  $\Omega = [0, 1] \times \bar{B}_{H^0}(\theta, \epsilon) \times \bar{B}_{H^0}(\theta, \epsilon) \times \partial \overline{\mathcal{Q}_{r,s}}$ .

*Proof.* Note that  $\partial \overline{\mathcal{Q}_{r,s}}$  is union of two closed subsets of it, i.e.

$$\partial \overline{\mathcal{Q}_{r,s}} = [(\partial B_{H^+}(\theta, r)) \oplus \bar{B}_{H^-}(\theta, s)] \cup [\bar{B}_{H^+}(\theta, r) \oplus (\partial B_{H^-}(\theta, s))].$$

Then  $\Omega = \Lambda_1 \cup \Lambda_2$ , where  $\Lambda_1 = [0, 1] \times \bar{B}_{H^0}(\theta, \epsilon) \times \bar{B}_{H^0}(\theta, \epsilon) \times (\partial B_{H^+}(\theta, r)) \oplus \bar{B}_{H^-}(\theta, s)$  and  $\Lambda_2 = [0, 1] \times \bar{B}_{H^0}(\theta, \epsilon) \times \bar{B}_{H^0}(\theta, \epsilon) \times B_{H^+}(\theta, r) \oplus (\partial \bar{B}_{H^-}(\theta, s))$ . We firstly prove

$$\inf\{\|tP^\perp \nabla \mathcal{L}(z_1 + u) + (1-t)P^\perp \nabla \mathcal{L}(z_2 + u)\| \mid (t, z_1, z_2, u) \in \Lambda_1\} > 0. \quad (2.14)$$

By a contradiction we assume that there exist sequences  $\{t_n\}_{n \geq 1} \subset [0, 1]$  and

$$\{z_n\}_{n \geq 1}, \{z'_n\}_{n \geq 1} \subset \bar{B}_{H^0}(\theta, \epsilon), \quad \{u_n\}_{n \geq 1} \subset (\partial B_{H^+}(\theta, r)) \oplus \bar{B}_{H^-}(\theta, s)$$

such that  $\|t_n P^\perp \nabla \mathcal{L}(z_n + u_n) + (1-t_n)P^\perp \nabla \mathcal{L}(z'_n + u_n)\| \rightarrow 0$ . Hence after removing finite many terms we can assume

$$(t_n P^\perp \nabla \mathcal{L}(z_n + u_n) + (1-t_n)P^\perp \nabla \mathcal{L}(z'_n + u_n), u_n^+)_H \leq \frac{a'_1 r^2}{4}, \quad \forall n \in \mathbb{N}, \quad (2.15)$$

$$(t_n P^\perp \nabla \mathcal{L}(z_n + u_n) + (1-t_n)P^\perp \nabla \mathcal{L}(z'_n + u_n), u_n^-)_H \geq -\frac{a'_1 r^2}{4}, \quad \forall n \in \mathbb{N}. \quad (2.16)$$

Note that  $u_n^+ \in \partial B_{H^+}(\theta, r)$  and  $u_n^- \in \bar{B}_{H^-}(\theta, s)$ . So (2.15) and (2.12) lead to

$$\frac{a'_1}{4} r^2 \geq (t_n P^\perp \nabla \mathcal{L}(z_n + u_n) + (1-t_n)P^\perp \nabla \mathcal{L}(z'_n + u_n), u_n^+)_H \geq a'_1 r^2 - \frac{a'_1}{2} \|u_n^-\|^2$$

and therefore

$$\frac{r^2}{\|u_n^-\|^2} \leq \frac{2}{3}, \quad \forall n \in \mathbb{N}. \quad (2.17)$$

Moreover, from (2.13) and (2.16) we derive

$$-\frac{a'_1}{4} r^2 \leq (t_n P^\perp \nabla \mathcal{L}(z_n + u_n) + (1-t_n)P^\perp \nabla \mathcal{L}(z'_n + u_n), u_n^-)_H \leq -a'_1 \|u_n^-\|^2 + \frac{a'_1}{2} r^2$$

and hence

$$\frac{r^2}{\|u_n^-\|^2} \geq \frac{4}{3}, \quad \forall n \in \mathbb{N},$$



which contradicts (2.17). (2.14) is proved.

Similarly, suppose that there exist sequences  $\{t_n\}_{n \geq 1} \subset [0, 1]$  and

$$\{z_n\}_n, \{z'_n\}_n \subset \bar{B}_{H^0}(\theta, \epsilon), \quad \{v_n\}_n \subset B_{H^+}(\theta, r) \oplus (\partial B_{H^-}(\theta, s))$$

such that  $\|t_n P^\perp \nabla \mathcal{L}(z_n + v_n) + (1 - t_n) P^\perp \nabla \mathcal{L}(z'_n + v_n)\| \rightarrow 0$ . As above we can assume

$$(t_n P^\perp \nabla \mathcal{L}(z_n + v_n) + (1 - t_n) P^\perp \nabla \mathcal{L}(z'_n + v_n), v_n^+)_H \leq \frac{a'_1 s^2}{4}, \quad \forall n \in \mathbb{N}, \quad (2.18)$$

$$(t_n P^\perp \nabla \mathcal{L}(z_n + v_n) + (1 - t_n) P^\perp \nabla \mathcal{L}(z'_n + v_n), v_n^-)_H \geq -\frac{a'_1 s^2}{4}, \quad \forall n \in \mathbb{N}. \quad (2.19)$$

Note that  $v_n^+ \in B_{H^+}(\theta, r)$  and  $v_n^- \in \partial B_{H^-}(\theta, s)$  for all  $n \in \mathbb{N}$ . Then (2.13) and (2.19) imply

$$-\frac{a'_1 s^2}{4} \leq (t_n P^\perp \nabla \mathcal{L}(z_n + v_n) + (1 - t_n) P^\perp \nabla \mathcal{L}(z'_n + v_n), v_n^-)_H \leq -a'_1 s^2 + \frac{a'_1}{2} \|v_n^+\|^2$$

and so

$$\frac{s^2}{\|v_n^+\|^2} \leq \frac{2}{3}, \quad \forall n \in \mathbb{N}. \quad (2.20)$$

With the same methods, (2.12) and (2.18) lead to

$$\frac{a'_1 s^2}{4} \geq (t_n P^\perp \nabla \mathcal{L}(z_n + v_n) + (1 - t_n) P^\perp \nabla \mathcal{L}(z'_n + v_n), v_n^+)_H \geq a'_1 \|v_n^+\|^2 - \frac{a'_1}{2} s^2$$

and so

$$\frac{s^2}{\|v_n^+\|^2} \geq \frac{4}{3}, \quad \forall n \in \mathbb{N}.$$

This contradicts (2.20). Hence

$$\inf\{\|t P^\perp \nabla \mathcal{L}(z_1 + u) + (1 - t) P^\perp \nabla \mathcal{L}(z_2 + u)\| \mid (t, z_1, z_2, u) \in \Lambda_2\} > 0.$$

This and (2.14) yield the desired conclusions.  $\square$

Since (D4) is equivalent to (D4\*), it was proved in [40, p. 2966–2967] that  $\nabla \mathcal{L}$  is of class  $(S)_+$  under the conditions (S), (F), (C) and (D) in [40]. In particular, this is also true under the assumptions of Theorem 2.1 (without requirement  $H^0 = \{\theta\}$ ), of course the conditions of Theorem 2.1 guarantee the same claim.

In the following we always assume that  $r > 0, s > 0$  and  $\epsilon > 0$  are as in Lemma 2.12.

**Lemma 2.13.** *For each  $z \in B_{H^0}(\theta, \epsilon)$ , the map*

$$f_z : \overline{\mathcal{Q}_{r,s}} \ni u \mapsto P^\perp \nabla \mathcal{L}(z + u) \in H^+ \oplus H^-$$

*is of class  $(S)_+$ . Moreover, for any two points  $z_0, z_1 \in B_{H^0}(\theta, \epsilon)$  the map  $\mathcal{H} : [0, 1] \times \overline{\mathcal{Q}_{r,s}} \rightarrow H^+ \oplus H^-$  given by*

$$\mathcal{H}(t, u) = (1 - t) P^\perp \nabla \mathcal{L}(z_0 + u) + t P^\perp \nabla \mathcal{L}(z_1 + u)$$

*is a homotopy of class  $(S)_+$  (cf. [52, Definition 4.40]).*

*Proof.* By [52, Proposition 4.41] we only need to prove the first claim. Let  $\{u_j\} \subset \overline{\mathcal{Q}_{r,s}}$  weakly converge to  $u \in H^+ \oplus H^-$ . Assume that they satisfy

$$\overline{\lim}(P^\perp \nabla \mathcal{L}(z + u_j), u_j - u)_H \leq 0.$$

It suffices to prove  $u_j \rightarrow u$  in  $H^+ \oplus H^-$ . Note that  $u_j \rightharpoonup u$  in  $H$  because  $\overline{\mathcal{Q}_{r,s}} \subset H^+ \oplus H^-$ . So is  $z + u_j \rightharpoonup z + u$  in  $H$ . Moreover,  $u_j - u \in H^+ \oplus H^-$  implies

$$\begin{aligned} (P^\perp \nabla \mathcal{L}(z + u_j), u_j - u)_H &= (\nabla \mathcal{L}(z + u_j), u_j - u)_H \\ &= (\nabla \mathcal{L}(z + u_j), (z + u_j) - (z + u))_H. \end{aligned}$$

It follows that  $\overline{\lim}(\nabla \mathcal{L}(z + u_j), (z + u_j) - (z + u))_H \leq 0$ . But  $\nabla \mathcal{L}$  is of class  $(S)_+$  near  $\theta \in H$ , we have  $z + u_j \rightarrow z + u$  and so  $u_j \rightarrow u$ .  $\square$

Let  $\deg$  denote the Browder-Skrypnik degree for demicontinuous  $(S)_+$ -maps ([8, 9], [58, 59, 60]), see [52, §4.3] for a nice exposition. By Lemma 2.12  $\deg(f_0, \mathcal{Q}_{r,s}, \theta)$  is well-defined and using the Poincaré-Hopf theorem (cf. [19, Theorem 1.2]) we have

$$\deg(f_0, \mathcal{Q}_{r,s}, \theta) = \sum_{q=0}^{\infty} (-1)^q \text{rank} C_q(f_0, \theta; G). \quad (2.21)$$

Note that  $\mathcal{L}|_{\mathcal{Q}_{r,s}}$  satisfies the conditions of Theorem 2.1. It follows that  $C_q(f_0, \theta; G) = \delta_{\mu q} G$ , where  $\mu = \dim H^-$ . Hence (2.21) becomes

$$\deg(f_0, \mathcal{Q}_{r,s}, \theta) = (-1)^\mu. \quad (2.22)$$

For each  $z \in B_{H^0}(\theta, \epsilon)$ , we derive from Lemma 2.12 that

$$\begin{aligned} \inf\{\|f_z(u)\| \mid u \in \partial \overline{\mathcal{Q}_{r,s}}\} &> 0 \quad \text{and} \\ \inf\{\|tf_z(u) + (1-t)f_0(u)\| \mid t \in [0, 1], u \in \partial \overline{\mathcal{Q}_{r,s}}\} &> 0. \end{aligned}$$

The former implies that  $\deg(f_z, \mathcal{Q}_{r,s}, \theta)$  is well-defined, the latter and Lemma 2.13 lead to

$$\deg(f_z, \mathcal{Q}_{r,s}, \theta) = \deg(f_0, \mathcal{Q}_{r,s}, \theta) = (-1)^\mu \quad (2.23)$$

by (2.22). So there exists a point  $u_z \in \mathcal{Q}_{r,s}$  such that

$$P^\perp \nabla \mathcal{L}(z + u_z) = f_z(u_z) = \theta. \quad (2.24)$$

Now let us give the main result in this subsection.

**Theorem 2.14** (Parameterized Implicit Function Theorem). *Under the assumptions of Theorem 2.2, suppose further that  $\mathcal{G}_1, \dots, \mathcal{G}_n \in C^1(V, \mathbb{R})$  satisfy*

- (i)  $\mathcal{G}'_j(\theta) = \theta, j = 1, \dots, n;$

(ii) for each  $j = 1, \dots, n$ , the gradient  $\nabla \mathcal{G}_j$  has Gâteaux derivative  $\mathcal{G}_j''(u) \in \mathcal{L}_s(H)$  at any  $u \in V$ , and  $\mathcal{G}_j'' : V \rightarrow \mathcal{L}_s(H)$  are continuous at  $\theta$ .

Then by shrinking  $\epsilon > 0$  (if necessary) we have  $\delta > 0$  and a unique continuous map

$$\psi : [-\delta, \delta]^n \times B_H(\theta, \epsilon) \cap H^0 \rightarrow \mathcal{Q}_{r,s} \subset (H^0)^\perp \quad (2.25)$$

such that for all  $(\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_H(\theta, \epsilon) \cap H^0$  with  $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ ,  $\psi(\vec{\lambda}, \theta) = \theta$  and

$$P^\perp \nabla \mathcal{L}(z + \psi(\vec{\lambda}, z)) + \sum_{j=1}^n \lambda_j P^\perp \nabla \mathcal{G}_j(z + \psi(\vec{\lambda}, z)) = \theta, \quad (2.26)$$

where  $P^\perp$  is as in (2.24). This  $\psi$  also satisfies

$$\|\psi(\vec{\lambda}, z_1) - \psi(\vec{\lambda}, z_2)\| \leq 3\|z_1 - z_2\|, \quad \forall (\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_H(\theta, \epsilon) \cap H^0. \quad (2.27)$$

Moreover, if  $G$  is a compact Lie group acting on  $H$  orthogonally,  $V$ ,  $\mathcal{L}$  and all  $\mathcal{G}_j$  are  $G$ -invariant (and hence  $H^0$ ,  $(H^0)^\perp$  are  $G$ -invariant subspaces, and  $\nabla \mathcal{L}$ ,  $\nabla \mathcal{G}_j$  are  $G$ -equivariant), then  $\psi$  is equivariant on  $z$ , i.e.,  $\psi(\vec{\lambda}, g \cdot z) = g \cdot \psi(\vec{\lambda}, z)$  for  $(\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_H(\theta, \epsilon) \cap H^0$  and  $g \in G$ .

**Proof. Step 1.** There exist  $\rho_1, \delta \in (0, 1)$  such that  $B_H(\theta, 2\rho_1) \subset V$  and that if sequences  $\vec{\lambda}_k = (\lambda_{k,1}, \dots, \lambda_{k,n}) \in [-\delta, \delta]^n$  converge to  $\vec{\lambda}_0 = (\lambda_{0,1}, \dots, \lambda_{0,n}) \in [-\delta, \delta]^n$ ,  $u_k \in B_H(\theta, 2\rho_1)$  weakly converge to  $u_0 \in B_H(\theta, 2\rho_1)$ , and they also satisfy

$$\overline{\lim}(\nabla \mathcal{L}(u_k) + \sum_{j=1}^n \lambda_j \nabla \mathcal{G}_j(u_k), u_k - u_0)_H \leq 0, \quad (2.28)$$

then  $u_k \rightarrow u_0$ . In particular, for each  $\vec{\lambda} \in [-\delta, \delta]^n$ , the map

$$B_H(\theta, 2\rho_1) \times [0, 1] \rightarrow H^+ \oplus H^-, \quad (t, u) \mapsto P^\perp \nabla \mathcal{L}(u) + \sum_{j=1}^n t \lambda_j P^\perp \nabla \mathcal{G}_j(u)$$

is a homotopy of class  $(S)_+$  (cf. [52, Definition 4.40]).

In fact, by [40, (5.8)] we had found  $\rho_1 > 0$  and  $C'_0 > 0$  such that  $B_H(\theta, 2\rho_1) \subset V$  and

$$\begin{aligned} (\nabla \mathcal{L}(u), u - u')_H &\geq \frac{C'_0}{2} \|u - u'\|^2 + (\nabla \mathcal{L}(u'), u - u')_H \\ &\quad + (Q(\theta)(u - u'), u - u')_H \end{aligned} \quad (2.29)$$

for any  $u, u' \in B_H(\theta, 2\rho_1)$ . Similarly, for each fixed  $j \in \{1, \dots, n\}$ , we have  $\tau = \tau(u, u') \in (0, 1)$  such that

$$\begin{aligned} &(\nabla \mathcal{G}_j(u), u - u')_H = (\nabla \mathcal{G}_j(u) - \nabla \mathcal{G}_j(u'), u - u')_H + (\nabla \mathcal{G}_j(u'), u - u')_H \\ &= (\mathcal{G}_j''(\tau u + (1 - \tau)u')(u - u'), u - u')_H + (\nabla \mathcal{G}_j(u'), u - u')_H \\ &= ([\mathcal{G}_j''(\tau u + (1 - \tau)u') - \mathcal{G}_j''(\theta)](u - u'), u - u')_H + (\nabla \mathcal{G}_j(u'), u - u')_H \\ &\quad + (\mathcal{G}_j''(\theta)(u - u'), u - u')_H, \quad \forall u, u' \in B_H(\theta, 2\rho_1). \end{aligned}$$

Since  $V \ni v \mapsto \mathcal{G}_j''(v) \in \mathcal{L}_s(H)$  is continuous at  $\theta$ , we may shrink  $\rho_1 > 0$  so that

$$\|\mathcal{G}_j''(v) - \mathcal{G}_j''(\theta)\| \leq \frac{C'_0}{8n}, \quad \forall v \in B_H(\theta, 2\rho_1), \quad j = 1, \dots, n. \quad (2.30)$$

It follows that for all  $u, u' \in B_H(\theta, 2\rho_1)$  and  $j = 1, \dots, n$ ,

$$\begin{aligned} |(\nabla \mathcal{G}_j(u), u - u')_H| &\leq \frac{C'_0}{8n} \|u - u'\|^2 + |(\nabla \mathcal{G}_j(u'), u - u')_H| \\ &\quad + |(\mathcal{G}_j''(\theta)(u - u'), u - u')_H|. \end{aligned}$$

Take  $\delta \in (0, 1)$  so that

$$\delta \sum_{j=1}^n \|\mathcal{G}_j''(\theta)\| < \frac{C'_0}{8}.$$

These and (2.29) imply that for all  $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in [-\delta, \delta]^n$ ,

$$\begin{aligned} &(\nabla \mathcal{L}(u), u - u')_H + \sum_{j=1}^n \lambda_j (\nabla \mathcal{G}_j(u), u - u')_H \\ &\geq \frac{C'_0}{4} \|u - u'\|^2 + (\nabla \mathcal{L}(u'), u - u')_H + (Q(\theta)(u - u'), u - u')_H \\ &\quad - \sum_{j=1}^n |(\nabla \mathcal{G}_j(u'), u - u')_H|. \end{aligned}$$

Replacing  $u, u'$  and  $\lambda_j$  by  $u_k, u_0$  and  $\lambda_{k,j}$  in the inequality, we derive from (2.28) that  $u_k \rightarrow u_0$  because (D3) implies that  $(\nabla \mathcal{L}(u_0), u_k - u_0)_H \rightarrow 0$ ,  $(Q(\theta)(u_k - u_0), u_k - u_0)_H \rightarrow 0$  and  $(\nabla \mathcal{G}_j(u_0), u_k - u_0)_H \rightarrow 0$ .

*Note:* The above proof shows that the family  $\{\mathcal{L}_{\vec{\lambda}} := \mathcal{L} + \sum_{j=1}^n \lambda_j \mathcal{G}_j \mid \vec{\lambda} \in [-\delta, \delta]^n\}$  satisfies the (PS) condition on  $\bar{B}_H(\theta, \varepsilon)$  for any  $\varepsilon < 2\rho_1$ , that is, if sequences  $\vec{\lambda}_k \in [-\delta, \delta]^n$  converge to  $\vec{\lambda}_0 \in [-\delta, \delta]^n$ , and  $u_k \in \bar{B}_H(\theta, \varepsilon)$  satisfies  $\nabla \mathcal{L}_{\vec{\lambda}_k}(u_k) \rightarrow \theta$  and  $\sup_k |\mathcal{L}_{\vec{\lambda}_k}(u_k)| < \infty$ , then  $\{u_k\}_{k \geq 1}$  has a converging subsequence  $u_{k_i} \rightarrow u_0 \in \bar{B}_H(\theta, \varepsilon)$  with  $\nabla \mathcal{L}_{\vec{\lambda}_0}(u_0) = \theta$ .

**Step 2.** Let  $r > 0, s > 0$  and  $\epsilon > 0$  be as in Lemma 2.12. By shrinking them, we can assume that  $\bar{B}_{H^0}(\theta, \epsilon) \times \overline{\mathcal{Q}_{r,s}} \subset B_H(\theta, 2\rho_1)$  and

$$\sup\{\|\nabla \mathcal{L}_{\vec{\lambda}}(z, u)\| \mid (\vec{\lambda}, z, u) \in [-1, 1]^n \times \bar{B}_{H^0}(\theta, \epsilon) \oplus \overline{\mathcal{Q}_{r,s}}\} < \infty \quad (2.31)$$

because  $\nabla \mathcal{L}$  and  $\nabla \mathcal{G}_1, \dots, \nabla \mathcal{G}_n$  are all locally bounded. Then by Lemma 2.12 we may shrink  $\delta \in (0, 1)$  so that

$$\inf \|tP^\perp(\nabla \mathcal{L} + \sum_{j=1}^n \lambda_j \nabla \mathcal{G}_j)(z_1 + u) + (1-t)P^\perp(\nabla \mathcal{L} + \sum_{j=1}^n \lambda_j \nabla \mathcal{G}_j)(z_2 + u)\| > 0,$$

where the infimum is taken for all  $(t, z_1, z_2, u) \in [0, 1] \times \bar{B}_{H^0}(\theta, \epsilon) \times \bar{B}_{H^0}(\theta, \epsilon) \times \partial \overline{\mathcal{Q}_{r,s}}$  and  $(\lambda_1, \dots, \lambda_n) \in [-\delta, \delta]^n$ . This implies that for each  $(\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon)$ , the map

$$f_{\vec{\lambda}, z} : \overline{\mathcal{Q}_{r,s}} \ni u \mapsto P^\perp \nabla \mathcal{L}(z + u) + \sum_{j=1}^n \lambda_j P^\perp \nabla \mathcal{G}_j(z + u) \in H^+ \oplus H^-$$

has a well-defined Browder-Skrypnik degree  $\deg(f_{\vec{\lambda},z}, \mathcal{Q}_{r,s}, \theta)$  and

$$\deg(f_{\vec{\lambda},z}, \mathcal{Q}_{r,s}, \theta) = \deg(f_{\vec{0},0}, \mathcal{Q}_{r,s}, \theta) = \deg(f_0, \mathcal{Q}_{r,s}, \theta) = (-1)^\mu, \quad (2.32)$$

where  $f_0$  is as in (2.22). Hence for each  $(\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon)$  there exists a point  $u_{\vec{\lambda},z} \in \mathcal{Q}_{r,s}$  such that

$$P^\perp \nabla \mathcal{L}(z + u_{\vec{\lambda},z}) + \sum_{j=1}^n \lambda_j P^\perp \nabla \mathcal{G}_j(z + u_{\vec{\lambda},z}) = f_{\vec{\lambda},z}(u_{\vec{\lambda},z}) = \theta. \quad (2.33)$$

By shrinking the above  $\epsilon > 0, r > 0$  and  $s > 0$  (if necessary) so that  $\omega$  and  $a_0, a_1$  in Lemma 2.8 can satisfy

$$\omega(z + u) < \min\{a_0, a_1\}/2, \quad \forall (z, u) \in \bar{B}_{H^0}(\theta, \epsilon) \times \overline{\mathcal{Q}_{r,s}}. \quad (2.34)$$

**Step 3.** If  $\delta > 0$  is sufficiently small, then  $u_{\vec{\lambda},z}$  is a unique zero point of  $f_{\vec{\lambda},z}$  in  $\mathcal{Q}_{r,s}$ .

In fact, suppose that there exists another different  $u'_{\vec{\lambda},z} \in \mathcal{Q}_{r,s}$  satisfying (2.33). We decompose

$u_{\vec{\lambda},z} - u'_{\vec{\lambda},z} = (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+ + (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-$ , and prove the conclusion in three cases:

- $\|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\| > \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\|$ ,
- $\|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\| = \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\|$ ,
- $\|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\| < \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\|$ .

Let us write  $\mathcal{L}_{\vec{\lambda}} = \mathcal{L} + \sum_{j=1}^n \lambda_j \mathcal{G}_j$  for conveniences. Then (2.33) implies

$$\begin{aligned} 0 &= (P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H \\ &= (P^\perp \nabla \mathcal{L}(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{L}(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H \\ &\quad + \sum_{j=1}^n \lambda_j (P^\perp \nabla \mathcal{G}_j(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{G}_j(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H. \end{aligned} \quad (2.35)$$

For simplicity we write  $u_{\vec{\lambda},z} = u_z$  and  $u'_{\vec{\lambda},z} = u'_z$ .

For the first two cases, we may use the mean value theorem to get  $\tau \in (0, 1)$  such that

$$\begin{aligned} &(P^\perp \nabla \mathcal{L}(z + u_z) - P^\perp \nabla \mathcal{L}(z + u'_z), (u_z - u'_z)^+)_H \\ &= (\nabla \mathcal{L}(z + u_z) - \nabla \mathcal{L}(z + u'_z), (u_z - u'_z)^+)_H \\ &= (B(z + \tau u_z + (1 - \tau)u'_z)(u_z - u'_z), (u_z - u'_z)^+)_H \\ &= (B(z + \tau u_z + (1 - \tau)u'_z)(u_z - u'_z)^+, (u_z - u'_z)^+)_H \\ &\quad + (B(z + \tau u_z + (1 - \tau)u'_z)(u_z - u'_z)^-, (u_z - u'_z)^+)_H \\ &\geq a_1 \|(u_z - u'_z)^+\|^2 - \omega(z + \tau u_z + (1 - \tau)u'_z) \|(u_z - u'_z)^-\| \cdot \|(u_z - u'_z)^+\| \\ &\geq a_1 \|(u_z - u'_z)^+\|^2 - \frac{a_1}{4} [\|(u_z - u'_z)^-\|^2 + \|(u_z - u'_z)^+\|^2] \\ &\geq a_1 \|(u_z - u'_z)^+\|^2 - \frac{a_1}{2} \|(u_z - u'_z)^+\|^2 \\ &= \frac{a_1}{2} \|(u_z - u'_z)^+\|^2, \end{aligned} \quad (2.36)$$

where the first inequality comes from Lemma 2.8(i)-(ii), the second one is derived from (2.34) and the inequality  $2|ab| \leq |a|^2 + |b|^2$ , and the third one is because  $\|(u_z - u'_z)^-\| \leq \|(u_z - u'_z)^+\|$ . It follows from (2.35)–(2.36) that

$$\begin{aligned} 0 &= (P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H \\ &\geq \frac{a_1}{2} \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|^2 \\ &\quad + \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \tau u_{\vec{\lambda},z} + (1 - \tau)u'_{\vec{\lambda},z})(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H. \end{aligned} \quad (2.37)$$

By (2.30) we have a constant  $M > 0$  such that

$$\sup\{\|\mathcal{G}_j''(z + w)\| \mid (z, w) \in \bar{B}_{H^0}(\theta, \epsilon) \times \overline{\mathcal{Q}_{r,s}}, j = 1, \dots, n\} < M. \quad (2.38)$$

Hence

$$\begin{aligned} &\sum_{j=1}^n |\lambda_j (\mathcal{G}_j''(z + \tau u_{\vec{\lambda},z} + (1 - \tau)u'_{\vec{\lambda},z})(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H| \\ &\leq n\delta M \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})\| \cdot \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\| \\ &\leq n\delta M [\|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|^2 + \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\| \cdot \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|] \\ &\leq 2n\delta M \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|^2. \end{aligned} \quad (2.39)$$

Let us shrink  $\delta > 0$  in Step 2 so that  $\delta < \frac{a_1}{8nM}$ . Then this and (2.37) lead to

$$0 = (P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+)_H \geq \frac{a_1}{4} \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|^2.$$

This contradicts  $(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+ \neq \theta$ .

Similarly, for the third case, as in (2.36) we may use Lemma 2.8(ii)-(iii) to obtain

$$\begin{aligned} 0 &= (P^\perp \nabla \mathcal{L}(z + u_z) - P^\perp \nabla \mathcal{L}(z + u'_z), (u_z - u'_z)^-)_H \\ &= (\nabla \mathcal{L}(z + u_z) - \nabla \mathcal{L}(z + u'_z), (u_z - u'_z)^-)_H \\ &= (B(z + tu_z + (1 - t)u'_z)(u_z - u'_z), (u_z - u'_z)^-)_H \\ &= (B(z + tu_z + (1 - t)u'_z)(u_z - u'_z)^-, (u_z - u'_z)^-)_H \\ &\quad + (B(z + tu_z + (1 - t)u'_z)(u_z - u'_z)^+, (u_z - u'_z)^-)_H \\ &\leq -a_0 \|(u_z - u'_z)^-\|^2 + \omega(z + tu_z + (1 - t)u'_z) \|(u_z - u'_z)^-\| \cdot \|(u_z - u'_z)^+\| \\ &\leq -a_0 \|(u_z - u'_z)^-\|^2 + \frac{a_0}{4} [\|(u_z - u'_z)^-\|^2 + \|(u_z - u'_z)^+\|^2] \\ &\leq -a_0 \|(u_z - u'_z)^-\|^2 + \frac{a_0}{2} \|(u_z - u'_z)^-\|^2 \\ &= -\frac{a_0}{2} \|(u_z - u'_z)^-\|^2, \end{aligned}$$

and hence

$$\begin{aligned}
0 &= (P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-)_H \\
&\leq -\frac{a_0}{2} \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|^2 \\
&\quad + \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \tau u_{\vec{\lambda},z} + (1-\tau)u'_{\vec{\lambda},z})(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-)_H. \quad (2.40)
\end{aligned}$$

As in (2.39) we may deduce

$$\begin{aligned}
&\sum_{j=1}^n |\lambda_j (\mathcal{G}_j''(z + \tau u_{\vec{\lambda},z} + (1-\tau)u'_{\vec{\lambda},z})(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-)_H| \\
&\leq n\delta M \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})\| \cdot \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\| \\
&\leq n\delta M [\|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\|^2 + \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\| \cdot \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^+\|] \\
&\leq 2n\delta M \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\|^2.
\end{aligned}$$

So if the above  $\delta > 0$  is also shrunk so that  $\delta < \frac{a_0}{8nM}$ , we may derive from this and (2.40) that

$$0 = (P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u_{\vec{\lambda},z}) - P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + u'_{\vec{\lambda},z}), (u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-)_H \leq -\frac{a_0}{4} \|(u_{\vec{\lambda},z} - u'_{\vec{\lambda},z})^-\|^2,$$

which also leads to a contradiction.

As a consequence, we have a well-defined map

$$\psi : [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon) \rightarrow \mathcal{Q}_{r,s}, \quad (\lambda, z) \mapsto u_{\vec{\lambda},z}. \quad (2.41)$$

**Step 4.**  $\psi$  is continuous.

Let sequence  $\{\vec{\lambda}_k\}_{k \geq 1} \in [-\delta, \delta]^n$  and  $\{z_k\}_{k \geq 1} \subset B_{H^0}(\theta, \epsilon)$  converge to  $\vec{\lambda}_0 \in [-\delta, \delta]^n$  and  $z_0 \in B_{H^0}(\theta, \epsilon)$ , respectively. We want to prove that  $\psi(\vec{\lambda}_k, z_k) \rightarrow \psi(\vec{\lambda}_0, z_0)$ . Since  $\{\psi(\vec{\lambda}_k, z_k)\}_{k \geq 1}$  is contained in  $\mathcal{Q}_{r,s}$ , we can suppose  $\psi(\vec{\lambda}_k, z_k) \rightharpoonup u_0 \in \overline{\mathcal{Q}_{r,s}}$  in  $H$ . Noting  $\psi(\vec{\lambda}_k, z_k) - u_0 \in H^+ \oplus H^-$ , by (2.33) we have

$$\begin{aligned}
&(\nabla \mathcal{L}_{\vec{\lambda}_k}(z_k + \psi(\vec{\lambda}_k, z_k)), \psi(\vec{\lambda}_k, z_k) - u_0) \\
&= (P^\perp \nabla \mathcal{L}_{\vec{\lambda}_k}(z_k + \psi(\vec{\lambda}_k, z_k)), \psi(\vec{\lambda}_k, z_k) - u_0) = 0.
\end{aligned}$$

It follows from this and (2.31) that

$$\begin{aligned}
&\|(\nabla \mathcal{L}_{\vec{\lambda}_k}(z_k + \psi(\vec{\lambda}_k, z_k)), (z_k + \psi(\vec{\lambda}_k, z_k)) - (z_0 + u_0))\| \\
&= \|(\nabla \mathcal{L}_{\vec{\lambda}_k}(z_k + \psi(\vec{\lambda}_k, z_k)), z_k - z_0)\| \\
&\leq \|\nabla \mathcal{L}_{\vec{\lambda}_k}(z_k + \psi(\vec{\lambda}_k, z_k))\| \cdot \|z_k - z_0\| \rightarrow 0.
\end{aligned}$$

As in the proof of Step 1, we may derive from this that  $z_k + \psi(\vec{\lambda}_k, z_k) \rightarrow z_0 + u_0$  and so  $\psi(\vec{\lambda}_k, z_k) \rightarrow u_0$ .

Moreover, (2.33) implies  $P^\perp \nabla \mathcal{L}_{\vec{\lambda}_k}(z_k + \psi(\vec{\lambda}_k, z_k)) = 0$ ,  $k = 1, 2, \dots$ . The  $C^1$ -smoothness of  $\mathcal{L}$  and all  $\mathcal{G}_j$  leads to  $P^\perp \nabla \mathcal{L}_{\vec{\lambda}_0}(z_0 + u_0) = 0$ . By Step 3 we arrive at  $\psi(\vec{\lambda}_0, z_0) = u_0$  and hence  $\psi$  is continuous at  $(\vec{\lambda}_0, z_0)$ .

**Step 5.** For any  $(\vec{\lambda}, z_i) \in [-\delta, \delta]^n \times B_H(\theta, \epsilon) \cap H^0$ ,  $i = 1, 2$ , by the definition of  $\psi$ , we have

$$P^\perp \nabla \mathcal{L}(z_i + \psi(\vec{\lambda}, z_i)) + \sum_{j=1}^n \lambda_j P^\perp \nabla \mathcal{G}_j(z_i + \psi(\vec{\lambda}, z_i)) = \theta, \quad i = 1, 2,$$

and hence for  $\Xi = z_1 - z_2 + \psi(\vec{\lambda}, z_1) - \psi(\vec{\lambda}, z_2)$  we derive

$$\begin{aligned} 0 &= (P^\perp \nabla \mathcal{L}(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^+)_H \\ &+ \sum_{j=1}^n \lambda_j (P^\perp \nabla \mathcal{G}_j(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{G}_j(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^+)_H. \end{aligned} \quad (2.42)$$

As in the proof of (2.36) we obtain  $\tau \in (0, 1)$  such that

$$\begin{aligned} &(P^\perp \nabla \mathcal{L}(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^+)_H \\ &= (B(\tau z_1 + \tau \psi(\vec{\lambda}, z_1) + (1 - \tau)z_2 + (1 - \tau)\psi(\vec{\lambda}, z_2)), \Xi^+)_H \\ &+ (B(\tau z_1 + \tau \psi(\vec{\lambda}, z_1) + (1 - \tau)z_2 + (1 - \tau)\psi(\vec{\lambda}, z_2)), (\Xi^0 + \Xi^-), \Xi^+)_H \\ &\geq a_1 \|\Xi^+\|^2 - \frac{a_1}{4} [\|\Xi^- + \Xi^0\|^2 + \|\Xi^+\|^2] \\ &= \frac{3a_1}{4} \|\Xi^+\|^2 - \frac{a_1}{4} \|\Xi^0\|^2 - \frac{a_1}{4} \|\Xi^-\|^2. \end{aligned} \quad (2.43)$$

Let us further shrink  $\delta > 0$  in Step 3 so that  $\delta < \frac{\min\{a_0, a_1\}}{16nM}$ . As in (2.39) we may deduce

$$\begin{aligned} &\sum_{j=1}^n \lambda_j (P^\perp \nabla \mathcal{G}_j(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{G}_j(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^+)_H \\ &\leq \sum_{j=1}^n |\lambda_j (\mathcal{G}_j''(\tau z_1 + (1 - \tau)z_2 + \tau \psi(\vec{\lambda}, z_1) + (1 - \tau)\psi(\vec{\lambda}, z_2)) \Xi, \Xi^+)_H| \\ &\leq n\delta M \|\Xi\| \cdot \|\Xi^+\| \leq 2n\delta M [\|\Xi\|^2 + \|\Xi^+\|^2] \\ &\leq \frac{a_1}{8} [\|\Xi^-\|^2 + \|\Xi^0\|^2 + 2\|\Xi^+\|^2]. \end{aligned} \quad (2.45)$$

This and (2.42)–(2.43) lead to

$$0 \geq \frac{3a_1}{4} \|\Xi^+\|^2 - \frac{a_1}{4} \|\Xi^0\|^2 - \frac{a_1}{4} \|\Xi^-\|^2 - \frac{a_1}{8} [\|\Xi^-\|^2 + \|\Xi^0\|^2 + 2\|\Xi^+\|^2]$$

and so

$$0 \geq 4\|\Xi^+\|^2 - 3\|\Xi^0\|^2 - 3\|\Xi^-\|^2. \quad (2.46)$$



Similarly, replacing  $\Xi^+$  by  $\Xi^-$  in (2.43) and (2.44) we derive

$$\begin{aligned}
& (P^\perp \nabla \mathcal{L}(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^-)_H \\
& \leq -\frac{3a_0}{4} \|\Xi^-\|^2 + \frac{a_0}{4} \|\Xi^0\|^2 + \frac{a_0}{4} \|\Xi^+\|^2, \\
& \quad \sum_{j=1}^n \lambda_j (P^\perp \nabla \mathcal{G}_j(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{G}_j(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^-)_H \\
& \leq \frac{a_0}{8} [\|\Xi^+\|^2 + \|\Xi^0\|^2 + 2\|\Xi^-\|^2].
\end{aligned}$$

As above these two inequalities and the equality

$$\begin{aligned}
0 &= (P^\perp \nabla \mathcal{L}(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^-)_H \\
&+ \sum_{j=1}^n \lambda_j (P^\perp \nabla \mathcal{G}_j(z_1 + \psi(\vec{\lambda}, z_1)) - P^\perp \nabla \mathcal{G}_j(z_2 + \psi(\vec{\lambda}, z_2)), \Xi^-)_H
\end{aligned}$$

yield:  $0 \geq 4\|\Xi^-\|^2 - 3\|\Xi^0\|^2 - 3\|\Xi^+\|^2$ . Combing with (2.46) we obtain

$$\|\Xi^+ + \Xi^-\|^2 = \|\Xi^+\|^2 + \|\Xi^-\|^2 \leq 6\|\Xi^0\|^2.$$

Note that  $\Xi^9 = z_1 - z_2$  and  $\Xi^+ + \Xi^- = \psi(\vec{\lambda}, z_1) - \psi(\vec{\lambda}, z_2)$ . The desired claim is proved.

**Step 6.** The uniqueness of  $\psi$  implies that it is equivariant on  $z$ .  $\square$

As a by-product we have also the following result though it is not used in this paper.

**Theorem 2.15** (Inverse Function Theorem). *If the assumptions of Theorem 2.1 hold with  $X = H$ , then  $\nabla \mathcal{L}$  is locally invertible at  $\theta$ .*

*Proof.* We can assume that  $\nabla \mathcal{L}$  is of class  $(S)_+$  in  $\overline{\mathcal{Q}_{r,s}}$ . Since  $H^0 = \{\theta\}$  and  $\nabla \mathcal{L} = f_0$ , we have

$$\deg(\nabla \mathcal{L}, \mathcal{Q}_{r,s}, \theta) = (-1)^\mu \quad (2.47)$$

by (2.22). Moreover,  $\varrho := \inf\{\|\nabla \mathcal{L}(u)\| \mid u \in \partial \overline{\mathcal{Q}_{r,s}}\} > 0$  by Lemma 2.12. For any given  $v \in B_H(\theta, \varrho)$ , let us define

$$\mathcal{H} : [0, 1] \times \overline{\mathcal{Q}_{r,s}} \rightarrow H, \quad (t, u) \mapsto \nabla \mathcal{L}(u) - tv.$$

Then  $\|\mathcal{H}(t, u)\| = \|\nabla \mathcal{L}(u) - tv\| \geq \|\nabla \mathcal{L}(u)\| - \|v\| \geq \varrho - \|v\| > 0$  for all  $(t, u) \in [0, 1] \times \partial \overline{\mathcal{Q}_{r,s}}$ . Assume that sequences  $t_n \rightarrow t$  in  $[0, 1]$ ,  $\{u_n\}_{n \geq 1} \subset \mathcal{Q}_{r,s}$  converges weakly to  $u$  in  $H$ , and they satisfy  $\limsup_{n \rightarrow \infty} (\mathcal{H}(t_n, u_n), u_n - u)_H \leq 0$ . Then

$$(\nabla \mathcal{L}(u_n), u_n - u)_H = (\mathcal{H}(t_n, u_n), u_n - u)_H + t_n(v, u_n - u)_H$$

leads to  $\limsup_{n \rightarrow \infty} (\nabla \mathcal{L}(u_n), u_n - u)_H \leq 0$ . It follows that  $u_n \rightarrow u$  in  $H$  because  $\nabla \mathcal{L}$  is of class  $(S)_+$  in  $\overline{\mathcal{Q}_{r,s}}$ . Hence  $\mathcal{H}$  is a homotopy of class  $(S)_+$ , and thus (2.47) gives

$$\deg(\nabla \mathcal{L} - v, \mathcal{Q}_{r,s}, \theta) = \deg(\nabla \mathcal{L}, \mathcal{Q}_{r,s}, \theta) = (-1)^\mu,$$

which implies  $\nabla \mathcal{L}(\xi_v) = v$  for some  $\xi_v \in \mathcal{Q}_{r,s}$ . By Step 3 in the proof of Theorem 2.15 (taking  $\vec{\lambda} = \vec{0}$ ) it is easily seen that the equation  $\nabla \mathcal{L}(u) = v$  has a unique solution in  $\overline{\mathcal{Q}_{r,s}}$ , and in particular,  $\xi_v$  is unique. Then we get a map  $B_H(\theta, \varrho) \rightarrow \mathcal{Q}_{r,s}$ ,  $v \mapsto \xi_v$  to satisfy  $\nabla \mathcal{L}(\xi_v) = v$  for all  $v \in B_H(\theta, \varrho)$ . We claim that this map is continuous. Arguing by contradiction, assume that there exists a sequence  $v_n \rightarrow v$  in  $B_H(\theta, \varrho)$ , such that  $\xi_{v_n} \rightharpoonup \xi^*$  in  $H$  and  $\|\xi_{v_n} - \xi_v\| \geq \epsilon_0$  for some  $\epsilon_0 > 0$  and all  $n = 1, 2, \dots$ . Note that

$$(\nabla \mathcal{L}(\xi_{v_n}), \xi_{v_n} - \xi^*)_H = (v_n, \xi_{v_n} - \xi^*)_H = (v_n - v, \xi_{v_n} - \xi^*)_H + (v, \xi_{v_n} - \xi^*)_H \rightarrow 0.$$

We derive that  $\xi_{v_n} \rightarrow \xi^*$  in  $H$ , and so  $\nabla \mathcal{L}(\xi_{v_n}) = v_n$  can lead to  $\nabla \mathcal{L}(\xi^*) = v$ . The uniqueness of solutions implies  $\xi^* = \xi_v$ . This prove the claim. Hence  $\nabla \mathcal{L}$  is a homeomorphism from an open neighborhood  $\{\xi_v \mid v \in B_H(\theta, \varrho)\}$  of  $\theta$  in  $\mathcal{Q}_{r,s}$  onto  $B_H(\theta, \varrho)$ .  $\square$

Theorem 2.15 cannot be derived from the invariance of domain theorem (5.4.1) of Berger [4] or [29, Theorem 2.5]. Recently, Ekeland proved an weaker inverse function theorem, [25, Theorem 2]. Since we cannot insure that  $B(u)$  has a right-inverse  $L(u)$  which is uniformly bounded in a neighborhood of  $\theta$ , [25, Theorem 2] it cannot lead to Theorem 2.15 either.

## 2.5 Parameterized splitting and shifting theorems

To shorten the proof of the main theorem, we shall write parts of it into two propositions.

**Proposition 2.16.** *Under the assumptions of Theorem 2.14, for each  $(\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon)$ , let  $\psi_{\vec{\lambda}}(z) = \psi(\vec{\lambda}, z)$  be given by (2.25). Then it satisfies*

$$\mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) = \min\{\mathcal{L}_{\vec{\lambda}}(z + u) \mid u \in B_H(\theta, r) \cap H^+\}$$

if  $H^- = \{\theta\}$ , and

$$\begin{aligned} \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) &= \min\{\mathcal{L}_{\vec{\lambda}}(z + u + P^-\psi_{\vec{\lambda}}(z)) \mid u \in B_H(\theta, r) \cap H^+\}, \\ \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) &= \max\{\mathcal{L}_{\vec{\lambda}}(z + P^+\psi_{\vec{\lambda}}(z) + v) \mid v \in B_H(\theta, s) \cap H^-\} \end{aligned}$$

if  $H^- \neq \{\theta\}$ .

*Proof.* **Case  $H^- = \{\theta\}$ .** We have  $\mathcal{Q}_{r,s} = B_H(\theta, r) \cap H^+$ , and (2.26) becomes

$$P^+ \nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) = 0, \quad \forall z \in B_{H^0}(\theta, \epsilon).$$

By this we may use integral mean value theorem to deduce that for each  $u \in \mathcal{Q}_{r,s}$ ,

$$\begin{aligned}
& \mathcal{L}_{\bar{\lambda}}(z+u) - \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)) \\
&= \int_0^1 (\nabla \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)+\tau(u-\psi_{\bar{\lambda}}(z))), u-\psi_{\bar{\lambda}}(z))_H d\tau \\
&= \int_0^1 (P^+ \nabla \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)+\tau(u-\psi_{\bar{\lambda}}(z))), u-\psi_{\bar{\lambda}}(z))_H d\tau \\
&= \int_0^1 (P^+ \nabla \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)+\tau(u-\psi_{\bar{\lambda}}(z))) - P^+ \nabla \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z))), u-\psi_{\bar{\lambda}}(z))_H d\tau \\
&= \int_0^1 (\nabla \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)+\tau(u-\psi_{\bar{\lambda}}(z))) - \nabla \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z))), u-\psi_{\bar{\lambda}}(z))_H d\tau \\
&= \int_0^1 (\nabla \mathcal{L}(z+\psi_{\bar{\lambda}}(z)+\tau(u-\psi_{\bar{\lambda}}(z))) - \nabla \mathcal{L}(z+\psi_{\bar{\lambda}}(z))), u-\psi_{\bar{\lambda}}(z))_H d\tau \\
&\quad + \sum_{j=1}^n \lambda_j \int_0^1 (\nabla \mathcal{G}_j(z+\psi_{\bar{\lambda}}(z)+\tau(u-\psi_{\bar{\lambda}}(z))) - \nabla \mathcal{G}_j(z+\psi_{\bar{\lambda}}(z))), u-\psi_{\bar{\lambda}}(z))_H d\tau \\
&= \int_0^1 \tau d\tau \int_0^1 (B(z+\psi_{\bar{\lambda}}(z)+\rho\tau(u-\psi_{\bar{\lambda}}(z)))(u-\psi_{\bar{\lambda}}(z)), u-\psi_{\bar{\lambda}}(z))_H d\rho \\
&\quad + \sum_{j=1}^n \lambda_j \int_0^1 \tau d\tau \int_0^1 (\mathcal{G}_j''(z+\psi_{\bar{\lambda}}(z)+\rho\tau(u-\psi_{\bar{\lambda}}(z)))(u-\psi_{\bar{\lambda}}(z)), u-\psi_{\bar{\lambda}}(z))_H d\rho \\
&\geq \frac{a_1}{2} \|u-\psi_{\bar{\lambda}}(z)\|^2 \\
&\quad + \sum_{j=1}^n \lambda_j \int_0^1 \tau d\tau \int_0^1 (\mathcal{G}_j''(z+\psi_{\bar{\lambda}}(z)+\rho\tau(u-\psi_{\bar{\lambda}}(z)))(u-\psi_{\bar{\lambda}}(z)), u-\psi_{\bar{\lambda}}(z))_H d\rho.
\end{aligned}$$

Here the final inequality comes from Lemma 2.8(i). For the final sum, as in (2.39) we have

$$\begin{aligned}
& \left| \sum_{j=1}^n \lambda_j \int_0^1 \tau d\tau \int_0^1 (\mathcal{G}_j''(z+\psi_{\bar{\lambda}}(z)+\rho\tau(u-\psi_{\bar{\lambda}}(z)))(u-\psi_{\bar{\lambda}}(z)), u-\psi_{\bar{\lambda}}(z))_H d\rho \right| \\
&\leq 2n\delta M \|u-\psi_{\bar{\lambda}}(z)\|^2 \leq \frac{a_1}{4} \|u-\psi_{\bar{\lambda}}(z)\|^2.
\end{aligned}$$

These lead to

$$\mathcal{L}_{\bar{\lambda}}(z+u) - \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)) \geq \frac{a_1}{4} \|u-\psi_{\bar{\lambda}}(z)\|^2, \quad (2.48)$$

which implies the desired conclusion.

**Case  $H^- \neq \{\theta\}$ .** For each  $u \in B_H(\theta, r) \cap H^+$  we have  $u + P^-\psi_{\bar{\lambda}}(z) \in \mathcal{Q}_{r,s}$ . As above we can use (2.26) to derive

$$\mathcal{L}_{\bar{\lambda}}(z+u+P^-\psi_{\bar{\lambda}}(z)) - \mathcal{L}_{\bar{\lambda}}(z+\psi_{\bar{\lambda}}(z)) \geq \frac{a_1}{4} \|u-P^+\psi_{\bar{\lambda}}(z)\|^2, \quad (2.49)$$

and therefore the second equality.

Finally, for each  $v \in B_H(\theta, r) \cap H^-$  we have  $P^+\psi_{\vec{\lambda}}(z) + v \in \mathcal{Q}_{r,s}$ . As above, using (2.26) and Lemma 2.8(ii)-(iii) we may deduce

$$\begin{aligned}
& \mathcal{L}_{\vec{\lambda}}(z + P^+\psi_{\vec{\lambda}}(z) + v) - \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) \\
&= \int_0^1 (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + t(u - P^+\psi_{\vec{\lambda}}(z))), v - P^-\psi_{\vec{\lambda}}(z))_H dt \\
&= \int_0^1 (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + t(v - P^-\psi_{\vec{\lambda}}(z))) - \nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)), v - P^-\psi_{\vec{\lambda}}(z))_H dt \\
&= \int_0^1 t \int_0^1 (B(z + \psi_{\vec{\lambda}}(z) + \tau t(v - P^-\psi_{\vec{\lambda}}(z)))(v - P^-\psi_{\vec{\lambda}}(z)), v - P^-\psi_{\vec{\lambda}}(z))_H d\tau dt \\
&+ \sum_{j=1}^n \lambda_j \int_0^1 t dt \int_0^1 (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + \tau t(v - P^-\psi_{\vec{\lambda}}(z)))(v - P^-\psi_{\vec{\lambda}}(z)), v - P^-\psi_{\vec{\lambda}}(z))_H d\tau \\
&\leq -\frac{a_0}{2} \|v - P^-\psi_{\vec{\lambda}}(z)\|^2 \\
&+ \sum_{j=1}^n \lambda_j \int_0^1 t dt \int_0^1 (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + \tau t(v - P^-\psi_{\vec{\lambda}}(z)))(v - P^-\psi_{\vec{\lambda}}(z)), v - P^-\psi_{\vec{\lambda}}(z))_H d\tau \\
&\leq -\frac{a_0}{4} \|v - P^-\psi_{\vec{\lambda}}(z)\|^2,
\end{aligned}$$

and hence the third equality.  $\square$

**Proposition 2.17.** *Under the assumptions of Theorem 2.14, for each  $(\vec{\lambda}, z) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon)$ , let  $\psi_{\vec{\lambda}}(z) = \psi(\vec{\lambda}, z)$  be given by (2.25). Then the functional  $\mathcal{L}_{\vec{\lambda}}^\circ : B_H(\theta, \epsilon) \cap H^0 \rightarrow \mathbb{R}$  given by*

$$\mathcal{L}_{\vec{\lambda}}^\circ(z) := \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) = \mathcal{L}(z + \psi(\vec{\lambda}, z)) + \sum_{j=1}^n \lambda_j \mathcal{G}_j(z + \psi(\vec{\lambda}, z)) \quad (2.50)$$

is of class  $C^1$ , and its differential is given by

$$\begin{aligned}
D\mathcal{L}_{\vec{\lambda}}^\circ(z)h &= D\mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z))h \\
&= D\mathcal{L}(z + \psi(\vec{\lambda}, z))h + \sum_{j=1}^n \lambda_j D\mathcal{G}_j(z + \psi(\vec{\lambda}, z))h, \quad \forall h \in H^0. \quad (2.51)
\end{aligned}$$

(Clearly, this implies that  $[-\delta, \delta]^n \ni \vec{\lambda} \mapsto \mathcal{L}_{\vec{\lambda}}^\circ \in C^1(\bar{B}_H(\theta, \epsilon) \cap H^0)$  is continuous by shrinking  $\epsilon > 0$  since  $\dim H^0 < \infty$ ).

*Proof.* **Case  $H^- \neq \{\theta\}$ .** For fixed  $z \in B_H(\theta, \epsilon) \cap H^0$ ,  $h \in H^0$ , and  $t \in \mathbb{R}$  with sufficiently small  $|t|$ , the last two equalities in Proposition 2.16 imply

$$\begin{aligned}
& \mathcal{L}_{\vec{\lambda}}(z + th + P^+\psi_{\vec{\lambda}}(z + th) + P^-\varphi(z)) - \mathcal{L}_{\vec{\lambda}}(z + P^+\psi_{\vec{\lambda}}(z + th) + P^-\psi_{\vec{\lambda}}(z)) \\
&\leq \mathcal{L}_{\vec{\lambda}}(z + th + \psi_{\vec{\lambda}}(z + th)) - \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)) \\
&\leq \mathcal{L}_{\vec{\lambda}}(z + th + P^+\psi_{\vec{\lambda}}(z) + P^-\psi_{\vec{\lambda}}(z + th)) - \mathcal{L}_{\vec{\lambda}}(z + P^+\psi_{\vec{\lambda}}(z) + P^-\psi_{\vec{\lambda}}(z + th)). \quad (2.52)
\end{aligned}$$

Since  $\mathcal{L}_{\bar{\lambda}}$  is  $C^1$  and  $\psi_{\bar{\lambda}}$  is continuous we deduce,

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\mathcal{L}_{\bar{\lambda}}(z + th + P^+ \psi_{\bar{\lambda}}(z + th) + P^- \psi_{\bar{\lambda}}(z)) - \mathcal{L}_{\bar{\lambda}}(z + P^+ \psi_{\bar{\lambda}}(z + th) + P^- \psi_{\bar{\lambda}}(z))}{t} \\ &= \lim_{t \rightarrow 0} \int_0^1 D\mathcal{L}_{\bar{\lambda}}(z + sth + P^+ \psi_{\bar{\lambda}}(z + th) + P^- \psi_{\bar{\lambda}}(z)) h ds \\ &= D\mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))h. \end{aligned} \quad (2.53)$$

Here the last equality follows from the Lebesgue's Dominated Convergence Theorem since

$$\{D\mathcal{L}_{\bar{\lambda}}(z + sth + P^+ \psi_{\bar{\lambda}}(z + th) + P^- \psi_{\bar{\lambda}}(z))h \mid 0 \leq s \leq 1, |t| \leq 1\}$$

is bounded by the compactness of  $\{z + sth + P^+ \psi_{\bar{\lambda}}(z + th) + P^- \psi_{\bar{\lambda}}(z) \mid 0 \leq s \leq 1, |t| \leq 1\}$ .

Similarly, we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\mathcal{L}_{\bar{\lambda}}(z + th + P^+ \psi_{\bar{\lambda}}(z) + P^- \psi_{\bar{\lambda}}(z + th)) - \mathcal{L}_{\bar{\lambda}}(z + P^+ \psi_{\bar{\lambda}}(z) + P^- \psi_{\bar{\lambda}}(z + th))}{t} \\ &= D\mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))h. \end{aligned} \quad (2.54)$$

Then it follows from (2.52)-(2.54) that

$$\lim_{t \rightarrow 0} \frac{\mathcal{L}_{\bar{\lambda}}(z + th + \psi_{\bar{\lambda}}(z + th)) - \mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))}{t} = D\mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))h.$$

That is,  $\mathcal{L}_{\bar{\lambda}}^\circ$  is Gâteaux differentiable and  $D\mathcal{L}_{\bar{\lambda}}^\circ(z) = D\mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))|_{H^0}$ . The latter implies that  $\mathcal{L}_{\bar{\lambda}}^\circ$  is of class  $C^1$  because both  $D\mathcal{L}_{\bar{\lambda}}$  and  $\psi_{\bar{\lambda}}$  are continuous.

**Case  $H^- = \{\theta\}$ .** For fixed  $z \in B_H(\theta, \epsilon) \cap H^0$  and  $h \in H^0$ , and  $t \in \mathbb{R}$  with sufficiently small  $|t|$ , the first equality in Proposition 2.16 implies

$$\begin{aligned} & \mathcal{L}_{\bar{\lambda}}(z + th + \psi_{\bar{\lambda}}(z + th)) - \mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z + th)) \\ &\leq \mathcal{L}_{\bar{\lambda}}(z + th + \psi_{\bar{\lambda}}(z + th)) - \mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z)) \\ &\leq \mathcal{L}_{\bar{\lambda}}(z + th + \psi_{\bar{\lambda}}(z)) - \mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z)). \end{aligned} \quad (2.55)$$

By the continuity of  $\nabla \mathcal{L}_{\bar{\lambda}}$  and  $\psi_{\bar{\lambda}}$  we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\mathcal{L}_{\bar{\lambda}}(z + th + \psi_{\bar{\lambda}}(z + th)) - \mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z + th))}{t} \\ &= \lim_{t \rightarrow 0} \int_0^1 D\mathcal{L}_{\bar{\lambda}}(z + sth + \psi_{\bar{\lambda}}(z + th)) h ds \\ &= D\mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))h. \end{aligned} \quad (2.56)$$

(As above this follows from the Lebesgue's Dominated Convergence Theorem because  $\{z + sth + \psi_{\bar{\lambda}}(z + th) \mid 0 \leq s \leq 1, 0 \leq t \leq 1\}$  is compact and thus  $\{\nabla \mathcal{L}_{\bar{\lambda}}(z + sth + \psi_{\bar{\lambda}}(z + th))h \mid 0 \leq s \leq 1, |t| \leq 1\}$  is bounded). Similarly, we may prove

$$\lim_{t \rightarrow 0} \frac{\mathcal{L}_{\bar{\lambda}}(z + th + \psi_{\bar{\lambda}}(z)) - \mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))}{t} = D\mathcal{L}_{\bar{\lambda}}(z + \psi_{\bar{\lambda}}(z))h, \quad (2.57)$$

and thus

$$\lim_{t \rightarrow 0} \frac{\mathcal{L}_{\vec{\lambda}}(z + th + \psi_{\vec{\lambda}}(z + th)) - \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z))}{t} = D\mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z))h$$

by (2.55)–(2.57). The desired claim follows immediately.  $\square$

**Theorem 2.18** (Parameterized Splitting Theorem). *Under the assumptions of Theorem 2.14, by shrinking  $\delta > 0$ ,  $\epsilon > 0$  and  $r > 0$ ,  $s > 0$ , we obtain an open neighborhood  $W$  of  $\theta$  in  $H$  and an origin-preserving homeomorphism*

$$\begin{aligned} [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, r) + B_{H^-}(\theta, s)) &\rightarrow [-\delta, \delta]^n \times W, \\ (\vec{\lambda}, z, u^+ + u^-) &\mapsto (\vec{\lambda}, \Phi_{\vec{\lambda}}(z, u^+ + u^-)) \end{aligned} \quad (2.58)$$

such that

$$\mathcal{L}_{\vec{\lambda}} \circ \Phi_{\vec{\lambda}}(z, u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 + \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z)) \quad (2.59)$$

for all  $(\vec{\lambda}, z, u^+ + u^-) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, r) + B_{H^-}(\theta, s))$ , where  $\psi$  is given by (2.25). The functional  $\mathcal{L}_{\vec{\lambda}}^\circ : B_H(\theta, \epsilon) \cap H^0 \rightarrow \mathbb{R}$  given by (2.50) is of class  $C^1$ , and its differential is given by (2.51). Moreover, (i) if  $\mathcal{L}$  and  $\mathcal{G}_j$ ,  $j = 1, \dots, n$ , are of class  $C^{2-0}$ , then so is  $\mathcal{L}_{\vec{\lambda}}^\circ$  for each  $\vec{\lambda} \in [-\delta, \delta]^n$ ; (ii) if a compact Lie group  $G$  acts on  $H$  orthogonally, and  $V$ ,  $\mathcal{L}$  and  $\mathcal{G}$  are  $G$ -invariant (and hence  $H^0$ ,  $(H^0)^\perp$  are  $G$ -invariant subspaces), then for each  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $\psi(\vec{\lambda}, \cdot)$  and  $\Phi_{\vec{\lambda}}(\cdot, \cdot)$  are  $G$ -equivariant, and  $\mathcal{L}_{\vec{\lambda}}^\circ(z) = \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z))$  is  $G$ -invariant.

Sometimes, for example, the corresponding conditions with [39, Theorem 1.1] or [40, Remark 3.2] are also satisfied, we can prove that  $\psi(\vec{\lambda}, \cdot)$  is of class  $C^1$  and that  $\mathcal{L}_{\vec{\lambda}}^\circ$  is of class  $C^2$ ; moreover,

$$D\mathcal{L}_{\vec{\lambda}}^\circ(z)u = (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z)), u)_H, \quad (2.60)$$

$$d^2\mathcal{L}_{\vec{\lambda}}^\circ(z)(u, v) = (\mathcal{L}_{\vec{\lambda}}''(z + \psi(\vec{\lambda}, z))(u + D_z\psi(\vec{\lambda}, z)u), v)_H \quad (2.61)$$

for all  $z \in B_H(\theta, \epsilon) \cap H^0$  and  $u, v \in H^0$ . In particular, since  $\psi(\vec{\lambda}, \theta) = \theta$  and  $D_z\psi(\vec{\lambda}, \theta) = \theta$ ,

$$d^2\mathcal{L}_{\vec{\lambda}}^\circ(\theta)(z_1, z_2) = (\mathcal{L}_{\vec{\lambda}}''(\theta)z_1, z_2)_H = - \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(\theta)z_1, z_2)_H, \quad \forall z_1, z_2 \in H^0. \quad (2.62)$$

**Claim.** *In this situation, if  $\theta \in H$  is a nondegenerate critical point of  $\mathcal{L}_{\vec{\lambda}}$  then  $\theta \in H^0$  such a critical point of  $\mathcal{L}_{\vec{\lambda}}^\circ$  too.*

In fact, suppose for some  $z_1 \in H^0$  that  $d^2\mathcal{L}_{\vec{\lambda}}^\circ(\theta)(z_1, z_2) = 0 \ \forall z_2 \in H^0$ . (2.62) implies  $(P^0\mathcal{L}_{\vec{\lambda}}''(\theta)z_1, u)_H = (P^0\mathcal{L}_{\vec{\lambda}}''(\theta)z_1, P^0u)_H = 0$  for all  $u \in H$ . Hence  $P^0\mathcal{L}_{\vec{\lambda}}''(\theta)z_1 = \theta$ . Moreover, since  $(I - P^0)\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z)) = \theta$  for all  $z \in B_H(\theta, \epsilon) \cap H^0$ . Differentiating this equality with respect to  $z$  we get  $(I - P^0)\mathcal{L}_{\vec{\lambda}}''(z + \psi(\vec{\lambda}, z))(u + D_z\psi(\vec{\lambda}, z)u) = \theta$  for all  $u \in H^0$ . In particular,  $(I - P^0)\mathcal{L}_{\vec{\lambda}}''(z)z = \theta$  for all  $z \in H^0$ . It follows that  $\mathcal{L}_{\vec{\lambda}}''(\theta)z_1 = \theta$  and hence  $z_1 = \theta$ .

*Proof of Theorem 2.18.* Let  $N = H^0$ , and for each  $\vec{\lambda} \in [-\delta, \delta]^n$  we define a map  $F_{\vec{\lambda}} : B_N(\theta, \epsilon) \times \mathcal{Q}_{r,s} \rightarrow \mathbb{R}$  by

$$F_{\vec{\lambda}}(z, u) = \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z) + u) - \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z)). \quad (2.63)$$

Then  $D_2 F_{\vec{\lambda}}(z, u)v = (P^\perp \nabla \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z) + u), v)_H$  for  $z \in \bar{B}_N(\theta, \epsilon)$ ,  $u \in \mathcal{Q}_{r,s}$  and  $v \in N^\perp$ . Moreover it holds that

$$F_{\vec{\lambda}}(z, \theta) = 0 \quad \text{and} \quad D_2 F_{\vec{\lambda}}(z, \theta)(v) = 0 \quad \forall v \in N^\perp. \quad (2.64)$$

Since  $B_N(\theta, \epsilon) \oplus \mathcal{Q}_{r,s}$  has the closure contained in the neighborhood  $U$  in Lemma 2.8, and  $\psi(\vec{\lambda}, \theta) = \theta$ , we can shrink  $\nu > 0$ ,  $\epsilon > 0$ ,  $r > 0$  and  $s > 0$  so small that

$$z + \psi(\vec{\lambda}, z) + u^+ + u^- \in U \quad (2.65)$$

for all  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $z \in \bar{B}_N(\theta, \epsilon)$  and  $u^+ + u^- \in \overline{\mathcal{Q}_{r,s}}$ .

Let us verify that each  $F_{\vec{\lambda}}$  satisfies conditions (ii)-(iv) in [40, Theorem A.1].

**Step 1.** For  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $z \in \bar{B}_N(\theta, \epsilon)$ ,  $u^+ \in \bar{B}_{H^+}(\theta, r)$  and  $u_1^-, u_2^- \in \bar{B}_{H^-}(\theta, \epsilon)$ , we have

$$\begin{aligned} & [D_2 F_{\vec{\lambda}}(z, u^+ + u_2^-) - D_2 F_{\vec{\lambda}}(z, u^+ + u_1^-)](u_2^- - u_1^-) \\ &= (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+ + u_2^-), u_2^- - u_1^-)_H \\ & \quad - (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+ + u_1^-), u_2^- - u_1^-)_H. \end{aligned} \quad (2.66)$$

Since  $\nabla \mathcal{L}_{\vec{\lambda}}$  is Gâteaux differentiable so is the function

$$u \mapsto (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+ + u), u_2^- - u_1^-)_H.$$

By the mean value theorem we have  $t \in (0, 1)$  such that

$$\begin{aligned} & (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+ + u_2^-), u_2^- - u_1^-)_H - (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+ + u_1^-), u_2^- - u_1^-)_H \\ &= (B(z + \psi_{\vec{\lambda}}(z) + u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-)_H \\ & \quad + \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-)_H \\ &\leq \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-)_H \\ & \quad - a_0 \|u_2^- - u_1^-\|^2 \end{aligned} \quad (2.67)$$

because of Lemma 2.8(iii). Recall that we have assumed  $\delta < \frac{\min\{a_0, a_1\}}{8nM}$  in Step 3 of the proof of Theorem 2.14. From this and (2.38) it follows that

$$\begin{aligned} & \sum_{j=1}^n |\lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + u^+ + u_1^- + t(u_2^- - u_1^-))(u_2^- - u_1^-), u_2^- - u_1^-)_H| \\ & \leq n\delta M \|u_2^- - u_1^-\|^2 \leq \frac{a_0}{8} \|u_2^- - u_1^-\|^2. \end{aligned}$$

This and (2.66)–(2.67) lead to

$$[D_2F_{\vec{\lambda}}(z, u^+ + u_2^-) - D_2F_{\vec{\lambda}}(z, u^+ + u_1^-)](u_2^- - u_1^-) \leq -\frac{a_0}{2}\|u_2^- - u_1^-\|^2.$$

This implies the condition (ii) of [40, theorem A.1].

**Step 2.** For  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $z \in \bar{B}_N(\theta, \epsilon)$ ,  $u^+ \in \bar{B}_{H^+}(\theta, r)$  and  $u^- \in \bar{B}_{H^-}(\theta, s)$ , by (2.64) and the mean value theorem, for some  $t \in (0, 1)$  we have

$$\begin{aligned} & D_2F_{\vec{\lambda}}(z, u^+ + u^-)(u^+ - u^-) \\ &= D_2F_{\vec{\lambda}}(z, u^+ + u^-)(u^+ - u^-) - D_2F_{\vec{\lambda}}(z, \theta)(u^+ - u^-) \\ &= (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+ + u^-), u^+ - u^-)_H - (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)), u^+ - u^-)_H \\ &= (B(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^-)_H \\ &+ \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^-)_H \\ &= (B(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))u^+, u^+)_H - (B(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))u^-, u^-)_H \\ &+ \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^-)_H \\ &\geq a_1\|u^+\|^2 + a_0\|u^-\|^2 \\ &+ \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^-)_H \end{aligned} \quad (2.68)$$

because of Lemma 2.8(i) and (iii). As above we have

$$\begin{aligned} & \sum_{j=1}^n |\lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^-)_H| \\ & \leq n\delta M\|u^+ + u^-\| \cdot \|u^+ - u^-\| \leq \frac{\min\{a_0, a_1\}}{4}(\|u^+\|^2 + \|u^-\|^2) \\ & \leq \frac{a_1}{4}\|u^+\|^2 + \frac{a_0}{4}\|u^-\|^2. \end{aligned}$$

From this and (2.68) we deduce

$$D_2F_{\vec{\lambda}}(z, u^+ + u^-)(u^+ - u^-) \geq \frac{a_1}{2}\|u^+\|^2 + \frac{a_0}{2}\|u^-\|^2.$$

The condition (iii) of [40, Theorem A.1] is satisfied.

**Step 3.** For  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $z \in \bar{B}_N(\theta, \epsilon)$  and  $u^+ \in \bar{B}_{H^+}(\theta, r)$ , as above we have  $t \in (0, 1)$  such that

$$\begin{aligned} & D_2F_{\vec{\lambda}}(z, u^+)u^+ = D_2F_{\vec{\lambda}}(z, u^+)u^+ - D_2F_{\vec{\lambda}}(z, \theta)u^+ \\ &= (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z) + u^+), u^+)_H - (\nabla \mathcal{L}_{\vec{\lambda}}(z + \psi_{\vec{\lambda}}(z)), u^+)_H \\ &= (B(z + \psi_{\vec{\lambda}}(z) + tu^+)u^+, u^+)_H + \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + tu^+)u^+, u^+)_H \\ &\geq a_1\|u^+\|^2 + \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + tu^+)u^+, u^+)_H \end{aligned}$$



because of Lemma 2.8(i). Moreover, it is proved as before that

$$\sum_{j=1}^n |\lambda_j (\mathcal{G}_j''(z + \psi_{\vec{\lambda}}(z) + tu^+)u^+, u^+)_H| \leq \frac{\min\{a_0, a_1\}}{8} \|u^+\|^2.$$

Hence we obtain

$$D_2 F_{\vec{\lambda}}(z, u^+)u^+ \geq a_1 \|u^+\|^2 > p(\|u^+\|) \quad \forall u^+ \in \bar{B}_{H^+}(\theta, s) \setminus \{\theta\},$$

where  $p : (0, \varepsilon] \rightarrow (0, \infty)$  is a non-decreasing function given by  $p(t) = \frac{a_1}{4}t^2$ . Namely,  $F_{\vec{\lambda}}$  satisfies the condition (iv) of [40, Theorem A.1] (the parameterized version of [23, Theorem 1.1]).

The other arguments are as before.

**Step 4.** The claim (i) in the part of “Moreover” follows from (2.27) directly. For the second one, since  $\psi(\lambda, \cdot)$  is  $G$ -equivariant, and  $\mathcal{L}_{\lambda}$  is  $G$ -invariant, we derive from (2.63) that  $F_{\vec{\lambda}}$  is  $G$ -invariant. By the construction of  $\Phi_{\vec{\lambda}}(\cdot, \cdot)$  (cf. [23] and [39, Theorem A.1]), it is expressed by  $F_{\vec{\lambda}}(z, \cdot)$ , one easily sees that  $\Phi_{\vec{\lambda}}(\cdot, \cdot)$  is  $G$ -equivariant.  $\square$

**Theorem 2.19** (Parameterized Shifting Theorem). *Suppose for some  $\vec{\lambda} \in [-\delta, \delta]^n$  that  $\theta \in H$  is an isolated critical point of  $\mathcal{L}_{\vec{\lambda}}$  (thus  $\theta \in H^0$  is that of  $\mathcal{L}_{\vec{\lambda}}^\circ$ ). Then*

$$C_q(\mathcal{L}_{\vec{\lambda}}, \theta; \mathbf{K}) = C_{q-\mu}(\mathcal{L}_{\vec{\lambda}}^\circ, \theta; \mathbf{K}) \quad \forall q \in \mathbb{N} \cup \{0\}, \quad (2.69)$$

where  $\mathcal{L}_{\vec{\lambda}}^\circ(z) = \mathcal{L}_{\vec{\lambda}}(z + \psi(\vec{\lambda}, z)) = \mathcal{L}(z + \psi(\vec{\lambda}, z)) + \sum_{j=1}^n \lambda_j \mathcal{G}_j(z + \psi(\vec{\lambda}, z))$  is as in (2.50).

*Proof.* Though  $\mathcal{L}_{\vec{\lambda}}$  and  $\mathcal{L}_{\vec{\lambda}}^\circ$  are only of class  $C^1$ , the construction of the Gromoll-Meyer pair on the pages 49-51 of [14] is also effective for them (see [16]). Hence the result can be obtained by repeating the proof of [14, Theorem I.5.4].

Of course, with a stability theorem of critical groups the present case can also be reduced to that of [14, Theorem I.5.4]. In fact, by Theorem 2.18 we have  $C_*(\mathcal{L}_{\vec{\lambda}}, \theta; \mathbf{K}) = C_*(\widehat{\mathcal{L}}_{\vec{\lambda}}, \theta; \mathbf{K})$ , where

$$\widehat{\mathcal{L}}_{\vec{\lambda}} : B_{H^0}(\theta, \epsilon) \times (H^+ \oplus H^-) \rightarrow \mathbb{R}, \quad (z, u^+ + u^-) \mapsto \|u^+\|^2 - \|u^-\|^2 + \mathcal{L}_{\vec{\lambda}}^\circ(z).$$

By a smooth cut function we can construct a  $C^1$  functional  $f : H^0 \rightarrow \mathbb{R}$  such that  $f(z) = \mathcal{L}_{\vec{\lambda}}^\circ(z)$  for  $\|z\| < \epsilon/2$ , and  $f(z) = \|z\|^2$  for  $\|z\| \geq 3\epsilon/4$ . Define a functional

$$\tilde{\mathcal{L}} : H^0 \times (H^+ \oplus H^-) \rightarrow \mathbb{R}, \quad (z, u^+ + u^-) \mapsto \|u^+\|^2 - \|u^-\|^2 + f(z).$$

Clearly,  $C_*(\widehat{\mathcal{L}}_{\vec{\lambda}}, \theta; \mathbf{K}) = C_*(\tilde{\mathcal{L}}, \theta; \mathbf{K})$ . Since  $\theta \in H^0$  is an isolated critical point of  $\mathcal{L}_{\vec{\lambda}}^\circ$ , by shrinking  $\epsilon > 0$  we can assume that  $\|\nabla \mathcal{L}_{\vec{\lambda}}^\circ(z)\| > 0$  for all  $z \in B_{H^0}(\theta, \epsilon) \setminus \{\theta\}$ . A standard result in differential topology claims that  $C^\infty(H^0, \mathbb{R})$  is dense in  $C_S^1(H^0, \mathbb{R})$  (equipped with strong topology). Hence we can choose a function  $g \in C^\infty(H^0, \mathbb{R})$  to satisfy

$$\begin{aligned} \|\nabla f(z) - \nabla g(z)\| &< \frac{1}{2} \|\nabla f(z)\|, \quad \forall z \in B_{H^0}(\theta, \epsilon) \setminus \{\theta\}, \\ \|f(z) - g(z)\| &< \frac{1}{2} \|f(z)\|, \quad \forall z \in H^0 \setminus B_{H^0}(\theta, 10\epsilon). \end{aligned}$$

They imply respectively that

$$\|\nabla((1-t)f + tg)(z)\| > \frac{1}{2}\|\nabla\mathcal{L}^\circ(z)\| > 0, \quad \forall z \in B_{H^0}(\theta, \epsilon/2) \setminus \{\theta\}, \quad (2.70)$$

$$\|(1-t)f(z) + tg(z)\| > \frac{1}{2}\|f(z)\| = \frac{1}{2}\|z\|^2, \quad \forall z \in H^0 \setminus B_{H^0}(\theta, 10\epsilon) \quad (2.71)$$

for all  $t \in [0, 1]$ . Hence each functional  $f_t : H^0 \rightarrow \mathbb{R}$ ,  $z \mapsto (1-t)f(z) + tg(z)$  has a unique critical point  $\theta$  in  $B_{H^0}(\theta, \epsilon/2)$  by (2.70), and satisfies (PS) condition by (2.71) and finiteness of  $\dim H^0$ . It follows from the stability theorem of critical groups ([22, Theorem 5.2]) that

$$C_*(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) = C_*(f, \theta; \mathbf{K}) = C_*(f_t, \theta; \mathbf{K}) = C_*(g, \theta; \mathbf{K}), \quad \forall t \in [0, 1]. \quad (2.72)$$

Since the functionals  $f_t$  and  $H^+ \oplus H^- \ni u^+ + u^- \mapsto \|u^+\|^2 - \|u^-\|^2 \in \mathbb{R}$  satisfies (PS) condition, so is each functional

$$\tilde{\mathcal{L}}_t : H^0 \times (H^+ \oplus H^-) \rightarrow \mathbb{R}, \quad (z, u^+ + u^-) \mapsto \|u^+\|^2 - \|u^-\|^2 + f_t(z).$$

As above we derive from [22, Theorem 5.2] that  $C_*(\hat{\mathcal{L}}, \theta; \mathbf{K}) = C_*(\tilde{\mathcal{L}}_1, \theta; \mathbf{K})$ . But [13, Theorem I.5.4] or [50, Theorem 8.4] implies  $C_*(\tilde{\mathcal{L}}_1, \theta; \mathbf{K}) = C_{*-\mu}(g, \theta; \mathbf{K})$ . Hence

$$C_*(\mathcal{L}_{\tilde{\lambda}}, \theta; \mathbf{K}) = C_*(\hat{\mathcal{L}}, \theta; \mathbf{K}) = C_*(\tilde{\mathcal{L}}_1, \theta; \mathbf{K}) = C_{*-\mu}(g, \theta; \mathbf{K}) = C_{*-\mu}(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}).$$

□

## 2.6 Splitting and shifting theorems around critical orbits

Let  $H$  be a Hilbert space with inner product  $(\cdot, \cdot)_H$  and let  $(\mathcal{H}, ((\cdot, \cdot)))$  be a  $C^3$  Hilbert-Riemannian manifold modeled on  $H$ . Let  $\mathcal{O} \subset \mathcal{H}$  be a compact  $C^3$  submanifold without boundary, and let  $\pi : N\mathcal{O} \rightarrow \mathcal{O}$  denote the normal bundle of it. The bundle is a  $C^2$ -Hilbert vector bundle over  $\mathcal{O}$ , and can be considered as a subbundle of  $T\mathcal{O}\mathcal{H}$  via the Riemannian metric  $((\cdot, \cdot))$ . The metric  $((\cdot, \cdot))$  induces a natural  $C^2$  orthogonal bundle projection  $\Pi : T\mathcal{O}\mathcal{H} \rightarrow N\mathcal{O}$ . For  $\varepsilon > 0$  we denote by

$$N\mathcal{O}(\varepsilon) := \{(x, v) \in N\mathcal{O} \mid \|v\|_x < \varepsilon\},$$

the so-called normal disk bundle of radius  $\varepsilon$ . If  $\varepsilon > 0$  is sufficiently small the exponential map  $\exp$  gives a  $C^2$ -diffeomorphism  $F$  from  $N\mathcal{O}(\varepsilon)$  onto an open neighborhood of  $\mathcal{O}$  in  $\mathcal{H}$ ,  $\mathcal{N}(\mathcal{O}, \varepsilon)$ .

For  $x \in \mathcal{O}$ , let  $\mathcal{L}_s(N\mathcal{O}_x)$  denote the space of those operators  $S \in \mathcal{L}(N\mathcal{O}_x)$  which are self-adjoint with respect to the inner product  $((\cdot, \cdot))_x$ , i.e.  $((S_x u, v))_x = ((u, S_x v))_x$  for all  $u, v \in N\mathcal{O}_x$ . Then we have a  $C^1$  vector bundle  $\mathcal{L}_s(N\mathcal{O}) \rightarrow \mathcal{O}$  whose fiber at  $x \in \mathcal{O}$  is given by  $\mathcal{L}_s(N\mathcal{O}_x)$ .

Let  $\mathcal{L} : \mathcal{H} \rightarrow \mathbb{R}$  be a  $C^1$  functional. A connected  $C^3$  submanifold  $\mathcal{O} \subset \mathcal{H}$  is called a *critical manifold* of  $\mathcal{L}$  if  $\mathcal{L}|_{\mathcal{O}} = \text{const}$  and  $D\mathcal{L}(x)v = 0$  for any  $x \in \mathcal{O}$  and  $v \in T_x\mathcal{H}$ . If there exists a neighborhood  $\mathcal{V}$  of  $\mathcal{O}$  such that  $\mathcal{V} \setminus \mathcal{O}$  contains no critical points of  $\mathcal{L}$  we say  $\mathcal{O}$  to be *isolated*.

Furthermore, we make the following

**Hypothesis 2.20.** The gradient field  $\nabla \mathcal{L} : \mathcal{H} \rightarrow T\mathcal{H}$  is Gâteaux differentiable and thus we have a bounded linear self-adjoint operator  $d^2\mathcal{L}(x) \in \mathcal{L}_s(T_x\mathcal{H})$  for each  $x \in \mathcal{O}$ ; moreover,  $\mathcal{O} \ni x \mapsto d^2\mathcal{L}(x)$  is a continuous section of  $\mathcal{L}_s(T\mathcal{H}) \rightarrow \mathcal{O}$  is continuous,  $\dim \text{Ker}(d^2\mathcal{L}(x)) = \text{const } \forall x \in \mathcal{O}$ , and there exists  $a_0 > 0$  such that

$$\sigma(d^2\mathcal{L}(x)) \cap ([-2a_0, 2a_0] \setminus \{0\}) = \emptyset \quad \forall x \in \mathcal{O}. \quad (2.73)$$

This implies that  $\mathcal{O} \ni x \mapsto \mathcal{B}_x(\theta_x) := \Pi_x \circ d^2\mathcal{L}(x)|_{N\mathcal{O}_x} = d^2(\mathcal{L} \circ \exp_x|_{N\mathcal{O}_x})(\theta_x)$  is a continuous section of  $\mathcal{L}_s(N\mathcal{O} \rightarrow \mathcal{O}, \dim \text{Ker}(\mathcal{B}_x(\theta_x)) = \text{const } \forall x \in \mathcal{O}$ , and

$$\sigma(\mathcal{B}_x(\theta_x)) \cap ([-2a_0, 2a_0] \setminus \{0\}) = \emptyset \quad \forall x \in \mathcal{O}.$$

Let  $\chi_*$  ( $*$  = +, −, 0) be the characteristic function of the intervals  $[2a_0, +\infty)$ ,  $(-2a_0, a_0)$  and  $(-\infty, -2a_0]$ , respectively. Then we have the orthogonal bundle projections on the normal bundle  $N\mathcal{O}$ ,  $P^*$  (defined by  $P_x^*(v) = \chi_*(\mathcal{B}_x(\theta_x))v$ ),  $*$  = +, −, 0. Denote by  $N^*\mathcal{O} = P^*N\mathcal{O}$ ,  $*$  = +, −, 0. (Clearly,  $\mathcal{B}_x(\theta_x)(N^*\mathcal{O}_x) \subset N^*\mathcal{O}_x$  for any  $x \in \mathcal{O}$  and  $*$  = +, −, 0). By [13, Lem.7.4], we have  $N\mathcal{O} = N^+\mathcal{O} \oplus N^-\mathcal{O} \oplus N^0\mathcal{O}$ . If  $\text{rank} N^0\mathcal{O} = 0$ , the critical orbit  $\mathcal{O}$  is called *nondegenerate*.

In the following we only consider the case  $\mathcal{O}$  is a critical orbit of a compact Lie group. The general case can be treated as in [42]. The following assumption implies naturally Hypothesis 2.20.

**Hypothesis 2.21. (i)** Let  $G$  be a compact Lie group, and let  $\mathcal{H}$  be a  $C^3$   $G$ -Hilbert manifold. (So  $T\mathcal{H}$  is a  $C^2$   $G$ -vector bundle, i.e. for any  $g \in G$  the induced action  $G \times T\mathcal{H} \rightarrow T\mathcal{H}$  given by  $g \cdot (x, v) = (g \cdot x, dg(x)v)$  is a  $C^1$  bundle map satisfying  $gT_x\mathcal{H} = T_{g \cdot x}\mathcal{H} \forall x \in \mathcal{H}$ ). Furthermore, this action also preserves the Riemannian-Hilbert structure  $((\cdot, \cdot))$ , i.e.

$$((g \cdot u, g \cdot v))_{g \cdot x} = ((u, v))_x, \quad \forall x \in \mathcal{H}, \quad \forall u, v \in T_x\mathcal{H}.$$

(In this case  $(\mathcal{H}, ((\cdot, \cdot)))$  is said to be a  $C^2$   $G$ -Riemannian-Hilbert manifold).

**(ii)** The  $C^1$  functional  $\mathcal{L} : \mathcal{H} \rightarrow \mathbb{R}$  is  $G$ -invariant,  $\nabla \mathcal{L} : \mathcal{H} \rightarrow T\mathcal{H}$  is Gâteaux differentiable, (i.e., under any  $C^3$  local chart the functional  $\mathcal{L}$  has a representation that is  $C^1$  and has a Gâteaux differentiable gradient map), and  $\mathcal{O}$  is an isolated critical orbit which is a  $C^3$  critical submanifold with Morse index  $\mu_{\mathcal{O}}$ .

Since  $\exp_{g \cdot x}(g \cdot v) = g \cdot \exp_x(v)$  for any  $g \in G$  and  $(x, v) \in T\mathcal{H}$ , we have

$$\mathcal{L} \circ \exp(g \cdot x, g \cdot v) = \mathcal{L}(\exp(g \cdot x, g \cdot v)) = \mathcal{L}(g \cdot \exp(x, v)) = \mathcal{L}(\exp(x, v)).$$

It follows that  $\nabla \mathcal{L}(g \cdot x) = g^{-1} \cdot \nabla \mathcal{L}(x)g$  and

$$\nabla (\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{gx}})(g \cdot v) = g \cdot \nabla (\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_x})(v) \quad (2.74)$$

for any  $g \in G$  and  $(x, v) \in N\mathcal{O}(\varepsilon)_x$ , which leads to

$$d^2 (\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{gx}})(g \cdot v) \cdot g = g \cdot d^2 (\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_x})(v) \quad (2.75)$$

as bounded linear operators from  $N\mathcal{O}_x$  onto  $N\mathcal{O}_{gx}$ .

**Theorem 2.22.** *Under Hypothesis 2.21, let for some  $x_0 \in \mathcal{O}$  the pair  $(\mathcal{L} \circ \exp_{x_0}, B_{T_{x_0}}\mathcal{H}(\theta, \varepsilon))$  (and so the pair  $(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}}, N\mathcal{O}(\varepsilon)_{x_0})$  by Lemma 2.10) satisfies the corresponding conditions in Hypothesis 1.1 with  $X = H$ . Suppose that the critical orbit  $\mathcal{O}$  is nondegenerate. Then there exist  $\epsilon > 0$  and a  $G$ -equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers,  $\Phi : N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow N\mathcal{O}$ , such that for any  $x \in \mathcal{O}$  and  $(u^+, u^-) \in N^+\mathcal{O}(\epsilon)_x \times N^-\mathcal{O}(\epsilon)_x$ ,*

$$\mathcal{L} \circ \exp \circ \Phi(x, u^+ + u^-) = \|u^+\|_x^2 - \|u^-\|_x^2 + \mathcal{L}|_{\mathcal{O}}. \quad (2.76)$$

It naturally leads to a Morse relation if  $\mathcal{L}$  satisfies the (PS) condition and has only nondegenerate critical orbits between regular levels. Theorem 2.22 will be proved after the proof of the following theorem.

**Theorem 2.23.** *Under Hypothesis 2.21, let for some  $x_0 \in \mathcal{O}$  the pair  $(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}}, N\mathcal{O}(\varepsilon)_{x_0})$  satisfy the corresponding conditions with Hypothesis 1.1 with  $X = H$ . Suppose that the critical orbit  $\mathcal{O}$  is degenerate, i.e.,  $\text{rank} N^0\mathcal{O} > 0$ . Then there exist  $\epsilon > 0$ , a  $G$ -equivariant topological bundle morphism that preserves the zero section,*

$$\mathfrak{h} : N^0\mathcal{O}(3\epsilon) \rightarrow N^+\mathcal{O} \oplus N^-\mathcal{O}, \quad (x, v) \mapsto \mathfrak{h}_x(v),$$

and a  $G$ -equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers,  $\Phi : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow N\mathcal{O}$ , such that the following properties hold:

(I) *The induced map,  $\mathfrak{h}_x : N^0\mathcal{O}(\epsilon)_x \rightarrow T_x\mathcal{H}$ , satisfies*

$$(P_x^+ + P_x^-) \circ \Pi_x \nabla(\mathcal{L} \circ \exp_x)(v + \mathfrak{h}_x(v)) = 0 \quad \forall (x, v^0) \in N^0\mathcal{O}(\epsilon). \quad (2.77)$$

(II)  *$\Phi$  has the form  $\Phi(x, v + u^+ + u^-) = (x, v + \mathfrak{h}_x(v) + \phi_{(x,v)}(u^+ + u^-))$  with  $\phi_{(x,v)}(u^+ + u^-) \in (N^+\mathcal{O} \oplus N^-\mathcal{O})_x$ , and satisfies*

$$\mathcal{L} \circ \exp \circ \Phi(x, v, u^+ + u^-) = \|u^+\|_x^2 - \|u^-\|_x^2 + \mathcal{L} \circ \exp_x(v + \mathfrak{h}_x(v)) \quad (2.78)$$

for any  $x \in \mathcal{O}$  and  $(v, u^+, u^-) \in N^0\mathcal{O}(\epsilon)_x \times N^+\mathcal{O}(\epsilon)_x \times N^-\mathcal{O}(\epsilon)_x$ . Moreover

(II.1)  $\Phi(x, v) = (x, v + \mathfrak{h}_x(v)) \quad \forall v \in N^0\mathcal{O}(\epsilon)_x$ ,

(II.2)  $\phi_{(x,v)}(u^+ + u^-) \in N^-\mathcal{O}$  if and only if  $u^+ = \theta_x$ .

(III) *For each  $x \in \mathcal{O}$  the function*

$$N^0\mathcal{O}(\epsilon)_x \rightarrow \mathbb{R}, \quad v \mapsto \mathcal{L}_x^\circ(v) := \mathcal{L} \circ \exp_x(v + \mathfrak{h}_x(v)) \quad (2.79)$$

is  $G_x$ -invariant, of class  $C^1$ , and satisfies

$$D\mathcal{L}_x^\circ(v)v' := (\nabla(\mathcal{L} \circ \exp_x)(v + \mathfrak{h}_x(v)), v'), \quad \forall v' \in N^0\mathcal{O}_x.$$

Moreover, each  $\mathfrak{h}_x$  is of class  $C^{1-0}$ , and hence  $\mathcal{L}_x^\circ$  is of class  $C^{2-0}$  if  $\mathcal{L}$  is of class  $C^{2-0}$ .

*Proof.* We only outline main procedures. By the assumption and (2.75) we deduce that each pair  $(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_x}, N\mathcal{O}(\varepsilon)_x)$  satisfies the corresponding conditions with Hypothesis 1.1 with  $X = H$  too, and that there exists  $a_0 > 0$  such that

$$\sigma(d^2(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_x})(\theta_x)) \cap ([-2a_0, 2a_0] \setminus \{0\}) = \emptyset, \quad \forall x \in \mathcal{O}. \quad (2.80)$$

By Theorem 2.2 we get a small  $\varepsilon \in (0, \varepsilon/3)$  and a continuous map

$$\mathfrak{h}_{x_0} : N^0\mathcal{O}(3\varepsilon)_{x_0} \rightarrow N^\pm\mathcal{O}(\varepsilon/2)_{x_0},$$

such that  $\mathfrak{h}_{x_0}(g \cdot v) = g \cdot \mathfrak{h}_{x_0}(v)$ ,  $\mathfrak{h}_{x_0}(\theta_{x_0}) = \theta_{x_0}$  and

$$(P_{x_0}^+ + P_{x_0}^-)\nabla(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}})(v + \mathfrak{h}_{x_0}(v)) = 0, \quad \forall v \in N^0\mathcal{O}(3\varepsilon)_{x_0}.$$

Furthermore, the function

$$\mathcal{L}_{x_0}^\circ : N^0\mathcal{O}(\varepsilon)_{x_0} \rightarrow \mathbb{R}, \quad v \mapsto \mathcal{L} \circ \exp_{x_0}(v + \mathfrak{h}_{x_0}(v))$$

is of class  $C^1$ , and  $D\mathcal{L}_{x_0}^\circ(v)u = (\nabla(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}})(v + \mathfrak{h}_{x_0}(v)), u)$ . Define

$$\mathfrak{h} : N^0\mathcal{O}(3\varepsilon) \rightarrow T\mathcal{H}, \quad (x, v) \mapsto g \cdot \mathfrak{h}_{x_0}(g^{-1} \cdot v),$$

where  $g \cdot x_0 = x$ . It is continuous. Otherwise, there exists a sequence  $\{(x_j, v_j)\}_j \subset N^0\mathcal{O}(3\varepsilon)$  which converges to a point  $(\bar{x}, \bar{v}) \in N^0\mathcal{O}(3\varepsilon)$ , but  $\mathfrak{h}(x_j, v_j) \not\rightarrow \mathfrak{h}(\bar{x}, \bar{v})$ . Passing to a subsequence we may assume that  $\{\mathfrak{h}(x_j, v_j)\}_j$  has no intersection with an open neighborhood  $\mathbf{U}$  of  $\mathfrak{h}(\bar{x}, \bar{v})$  in  $T\mathcal{H}$ . Let  $\bar{g}, g_j \in G$  be such that  $\bar{g} \cdot x_0 = \bar{x}$  and  $g_j \cdot x_0 = x_j$ ,  $j = 1, 2, \dots$ . Then  $\mathfrak{h}(\bar{x}, \bar{v}) = \bar{g} \cdot \mathfrak{h}_{x_0}(\bar{g}^{-1} \cdot \bar{v})$  and  $\mathfrak{h}(x_j, v_j) = g_j \cdot \mathfrak{h}_{x_0}(g_j^{-1} \cdot v_j)$  for each  $j \in \mathbb{N}$ . Note that  $\bar{g}^{-1} \cdot \mathbf{U}$  is an open neighborhood of  $\mathfrak{h}_{x_0}(\bar{g}^{-1} \cdot \bar{v}) = \bar{g}^{-1} \cdot \mathfrak{h}(\bar{x}, \bar{v})$  and that  $\{\bar{g}^{-1} \cdot \mathfrak{h}(x_j, v_j) = \bar{g}^{-1} \cdot g_j \cdot \mathfrak{h}_{x_0}(g_j^{-1} \cdot v_j)\}_j$  has no intersection with  $\bar{g}^{-1} \cdot \mathbf{U}$ . Since  $G$  is compact, we may assume  $\bar{g}^{-1} \cdot g_j \rightarrow \hat{g} \in G$  and so  $g_j^{-1} \rightarrow (\bar{g}\hat{g})^{-1} \in G$  after passing to a subsequence (if necessary). Then  $\bar{g}^{-1} \cdot \mathfrak{h}(x_j, v_j) = \bar{g}^{-1} \cdot g_j \cdot \mathfrak{h}_{x_0}(g_j^{-1} \cdot v_j) \rightarrow \hat{g} \cdot \mathfrak{h}_{x_0}((\bar{g}\hat{g})^{-1} \cdot \bar{v}) = \mathfrak{h}_{x_0}(\bar{g}^{-1} \cdot \bar{v})$ . It follows that  $\mathfrak{h}_{x_0}(\bar{g}^{-1} \cdot \bar{v})$  does not belong to  $\bar{g}^{-1} \cdot \mathbf{U}$ . This contradicts the fact that  $\bar{g}^{-1} \cdot \mathbf{U}$  is an open neighborhood of  $\mathfrak{h}_{x_0}(\bar{g}^{-1} \cdot \bar{v})$ .

By the definition of  $\mathfrak{h}$ , it is clearly  $G$ -equivariant and satisfies

$$(P_x^+ + P_x^-)\nabla(\mathcal{L} \circ \exp|_{N\mathcal{O}_x(\varepsilon)})(v + \mathfrak{h}_x(v)) = 0, \quad \forall (x, v) \in N^0\mathcal{O}(3\varepsilon). \quad (2.81)$$

Moreover, the map  $\mathcal{F} : N^0\mathcal{O}(\varepsilon) \oplus N^+\mathcal{O}(\varepsilon) \oplus N^-\mathcal{O}(\varepsilon) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \mathcal{F}(x, v, u^+ + u^-) &= \mathcal{F}_x(v, u^+ + u^-) \\ &= \mathcal{L} \circ \exp_x(v + \mathfrak{h}_x(v) + u^+ + u^-) - \mathcal{L} \circ \exp_x(v + \mathfrak{h}_x(v)), \end{aligned} \quad (2.82)$$

is  $G$ -invariant, and satisfies for any  $(x, v) \in N^0\mathcal{O}(\varepsilon)$  and  $u \in N^+\mathcal{O}_x \oplus N^-\mathcal{O}_x$ ,

$$\mathcal{F}_x(v, \theta_x) = 0 \quad \text{and} \quad D_2\mathcal{F}_x(v, \theta_x)[u] = 0. \quad (2.83)$$

By (2.74)–(2.75) and Lemmas 2.7, 2.8 we can immediately obtain:

**Lemma 2.24.** *There exist positive numbers  $\varepsilon_1 \in (0, \varepsilon)$  and  $a_1 \in (0, 2a_0)$ , and a function  $\Omega : N\mathcal{O}(\varepsilon_1) \rightarrow [0, \infty)$  with the property that  $\Omega(x, v) \rightarrow 0$  as  $\|v\|_x \rightarrow 0$ , such that for any  $(x, v) \in N\mathcal{O}(\varepsilon_1)$  the following conclusions hold with  $\mathcal{B}_x = d^2(\mathcal{L} \circ \exp|_{N\mathcal{O}_x(\varepsilon)})$ :*

- (i)  $|((\mathcal{B}_x(v)u, w))_x - ((\mathcal{B}_x(\theta_x)u, w))_x| \leq \Omega(x, v)\|u\|_x \cdot \|w\|_x$  for any  $u \in N^0\mathcal{O}_x \oplus N^-\mathcal{O}_x$  and  $w \in N\mathcal{O}_x$ ;
- (ii)  $((\mathcal{B}_x(v)u, u))_x \geq a_1\|u\|_x^2$  for all  $u \in N^+\mathcal{O}_x$ ;
- (iii)  $|((\mathcal{B}_x(v)u, w))_x| \leq \Omega(x, v)\|u\|_x \cdot \|w\|_x$  for all  $u^+ \in N^+\mathcal{O}_x, w \in N^-\mathcal{O}_x \oplus N^0\mathcal{O}_x$ ;
- (iv)  $((\mathcal{B}_x(v)u, u))_x \leq -a_0\|u\|_x^2$  for all  $u \in N^-\mathcal{O}_x$ .

Let us choose  $\varepsilon_2 \in (0, \varepsilon/2)$  so small that

$$(x, v^0 + \mathfrak{h}_x(v^0) + u^+ + u^-) \in N\mathcal{O}(\varepsilon_1)$$

for  $(x, v^0) \in N^0\mathcal{O}(2\varepsilon_2)$  and  $(x, u^*) \in N^*\mathcal{O}(2\varepsilon_2)$ ,  $*$  = +, -. As in the proof of [40, Lemma 3.5], we may use [40, Lemma 2.4] to derive

**Lemma 2.25.** *Let the constants  $a_1$  and  $a_0$  be given by Lemma 2.24(ii),(iv). For the above  $\varepsilon_2 > 0$  and each  $x \in \mathcal{O}$  the restriction of the functional  $\mathcal{F}_x$  to  $\overline{N^0\mathcal{O}(2\varepsilon_2)_x} \oplus [\overline{N^+\mathcal{O}(2\varepsilon_2)_x} \oplus \overline{N^-\mathcal{O}(2\varepsilon_2)_x}]$  satisfies:*

- (i)  $[D_2\mathcal{F}_x(v^0, u^+ + u^-) - D_2\mathcal{F}_x(v^0, u^+ + u^-)](u_2^- - u_1^-) \leq -a_1\|u_2^- - u_1^-\|_x^2$  for any  $(x, v^0) \in \overline{N^0\mathcal{O}(2\varepsilon_2)}$ ,  $(x, u^+) \in \overline{N^+\mathcal{O}(2\varepsilon_2)}$  and  $(x, u_j^-) \in \overline{N^-\mathcal{O}(2\varepsilon_2)}$ ,  $j = 1, 2$ ;
- (ii)  $D_2\mathcal{F}_x(v^0, u^+ + u^-)(u^+ - u^-) \geq a_1\|u^+\|_x^2 + a_0\|u^-\|_x^2$  for any  $(x, v^0) \in \overline{N^0\mathcal{O}(2\varepsilon_2)}$  and  $(x, u^*) \in \overline{N^*\mathcal{O}(2\varepsilon_2)}$ ,  $*$  = +, -;
- (iii)  $D_2\mathcal{F}_x(v^0, u^+)u^+ \geq a_1\|u^+\|_x^2$  for any  $(x, v^0) \in \overline{N^0\mathcal{O}(2\varepsilon_2)}$  and  $(x, u^+) \in \overline{N^+\mathcal{O}(2\varepsilon_2)}$ .

Denote by bundle projections  $\Pi_0 : \overline{N^0\mathcal{O}(\varepsilon_2)} \rightarrow \mathcal{O}$  and

$$\Pi_\pm : N^+\mathcal{O} \oplus N^-\mathcal{O} \rightarrow \mathcal{O}, \quad \Pi_* : N^*\mathcal{O} \rightarrow \mathcal{O}, \quad * = +, -.$$

Let  $\Lambda = \overline{N^0\mathcal{O}(2\varepsilon_2)}$ ,  $p : \mathcal{E} \rightarrow \Lambda$  and  $p_* : \mathcal{E}^* \rightarrow \Lambda$  be the pullbacks of  $N^+\mathcal{O} \oplus N^-\mathcal{O}$  and  $N^*\mathcal{O}$  via  $\Pi_0$ ,  $*$  = +, -. Then  $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ , and for  $\lambda = (x, v^0) \in \Lambda$  we have  $\mathcal{E}_\lambda = N^+\mathcal{O}_x \oplus N^-\mathcal{O}_x$  and  $\mathcal{E}_\lambda^* = N^*\mathcal{O}_x$ ,  $*$  = +, -. Moreover, for each  $\eta > 0$  we write

$$\begin{aligned} B_\eta(\mathcal{E}) &= \{(\lambda, w) \mid \lambda = (x, v^0) \in \Lambda \text{ \& } w \in (N^+\mathcal{O} \oplus N^-\mathcal{O})_x(\eta)\}, \\ \bar{B}_\eta(\mathcal{E}) &= \{(\lambda, w) \mid \lambda = (x, v^0) \in \Lambda \text{ \& } w \in \overline{(N^+\mathcal{O} \oplus N^-\mathcal{O})_x(\eta)}\}. \end{aligned}$$

Similarly,  $B_\eta(\mathcal{E}^*)$  and  $\bar{B}_\eta(\mathcal{E}^*)$  ( $*$  = +, -) are defined. Let  $\mathcal{J} : B_{2\varepsilon_2}(\mathcal{E}) \rightarrow \mathbb{R}$  be given by

$$\begin{aligned} \mathcal{J}(\lambda, v^\pm) &= \mathcal{J}_\lambda(v^\pm) = \mathcal{F}(x, v^0, v^\pm), \\ \forall \lambda &= (x, v^0) \in \Lambda \text{ \& } \forall v^\pm \in B_{2\varepsilon_2}(\mathcal{E})_\lambda. \end{aligned} \tag{2.84}$$

It is continuous, and  $C^1$  in  $v^\pm$ . From (2.83) and Lemma 2.25 we directly obtain

**Lemma 2.26.** *The functional  $\mathcal{J}_\lambda$  satisfies the conditions in Theorem A.2 of [40] (the bundle parameterized version of [23, Theorem 1.1]). Precisely, for each  $\lambda \in \Lambda$  the functional  $\mathcal{J}_\lambda : B_{2\varepsilon_2}(\mathcal{E}) \rightarrow \mathbb{R}$  satisfies:*

- (i)  $\mathcal{J}_\lambda(\theta_\lambda) = 0$  and  $D\mathcal{J}_\lambda(\theta_\lambda) = 0$ ;
- (ii)  $[D\mathcal{J}_\lambda(u^+ + u_2^-) - D\mathcal{J}_\lambda(u^+ + u_1^-)](u_2^- - u_1^-) \leq -a_1\|u_2^- - u_1^-\|_x^2$  for any  $\lambda = (x, v^0) \in \Lambda$ ,  $u^+ \in \bar{B}_{\varepsilon_2}(\mathcal{E}^+)_\lambda$  and  $u_j^- \in \bar{B}_{\varepsilon_2}(\mathcal{E}^-)_\lambda$ ,  $j = 1, 2$ ;
- (iii)  $D\mathcal{J}_\lambda(\lambda, u^+ + u^-)(u^+ - u^-) \geq a_1\|u^+\|_x^2 + a_0\|u^-\|_x^2$  for any  $\lambda = (x, v^0) \in \Lambda$  and  $u^* \in \bar{B}_{\varepsilon_2}(\mathcal{E}^*)_\lambda$ ,  $* = +, -$ ;
- (iv)  $D\mathcal{J}_\lambda(u^+)u^+ \geq a_1\|u^+\|_x^2$  for any  $\lambda = (x, v^0) \in \Lambda$  and  $u^+ \in \bar{B}_{\varepsilon_2}(\mathcal{E}^+)_\lambda$ .

By this we can use Theorem A.2 of [40] to get  $\epsilon \in (0, \varepsilon_2)$ , an open neighborhood  $U$  of the zero section  $0_{\mathcal{E}}$  of  $\mathcal{E}$  in  $B_{2\varepsilon_2}(\mathcal{E})$  and a homeomorphism

$$\phi : B_\epsilon(\mathcal{E}^+) \oplus B_\epsilon(\mathcal{E}^-) \rightarrow U, (\lambda, u^+ + u^-) \mapsto (\lambda, \phi_\lambda(u^+ + u^-)) \quad (2.85)$$

such that for all  $(\lambda, u^+ + u^-) \in B_\epsilon(\mathcal{E}^+) \oplus B_\epsilon(\mathcal{E}^-)$  with  $\lambda = (x, v^0) \in \Lambda$ ,

$$J(\phi(\lambda, u^+ + u^-)) = \|u^+\|_x^2 - \|u^-\|_x^2. \quad (2.86)$$

Moreover, for each  $\lambda \in \Lambda$ ,  $\phi_\lambda(\theta_\lambda) = \theta_\lambda$ ,  $\phi_\lambda(x + y) \in \mathcal{E}_\lambda^-$  if and only if  $x = \theta_\lambda$ , and  $\phi$  is a homeomorphism from  $B_\epsilon(\mathcal{E}^-)$  onto  $U \cap \mathcal{E}^-$ .

Note that  $B_\epsilon(\mathcal{E}^+) \oplus B_\epsilon(\mathcal{E}^-) = \overline{N^0\mathcal{O}(2\varepsilon_2)} \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon)$  and  $U = \overline{N^0\mathcal{O}(2\varepsilon_2)} \oplus \hat{U}$ , where  $\hat{U}$  is an open neighborhood of the zero section of  $N^+\mathcal{O} \oplus N^-\mathcal{O}$  in  $N^+\mathcal{O}(2\varepsilon_2) \oplus N^-\mathcal{O}(\varepsilon_2)$ . Let  $\mathcal{W} = N^0\mathcal{O}(\epsilon) \oplus \hat{U}$ , which is an open neighborhood of the zero section of  $N\mathcal{O}$  in  $N^0\mathcal{O}(2\varepsilon_2) \oplus N^+\mathcal{O}(2\varepsilon_2) \oplus N^-\mathcal{O}(\varepsilon_2)$ . By (2.85) we get a homeomorphism

$$\begin{aligned} \phi : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) &\rightarrow \mathcal{W}, \\ (x, v, u^+ + u^-) &\mapsto (x, v, \phi_{(x,v)}(u^+ + u^-)), \end{aligned} \quad (2.87)$$

and therefore a topological embedding bundle morphism that preserves the zero section,

$$\begin{aligned} \Phi : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) &\rightarrow N\mathcal{O}, \\ (x, v, u^+ + u^-) &\mapsto (x, v + \mathfrak{h}_x(v), \phi_{(x,v)}(u^+ + u^-)). \end{aligned} \quad (2.88)$$

From (2.82), (2.84) and (2.86) it follows that  $\Phi$  and  $\phi$  satisfy

$$\begin{aligned} \mathcal{L} \circ \exp \circ \Phi(x, v + u^+ + u^-) &= \mathcal{L} \circ \exp_x(v + \mathfrak{h}_x(v) + \phi_{(x,v)}(u^+ + u^-)) \\ &= \|u^+\|_x^2 - \|u^-\|_x^2 + \mathcal{L} \circ \exp_x(v + \mathfrak{h}_x(v)) \end{aligned}$$

for all  $(x, v + u^+, u^-) \in N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon)$ . The other conclusions easily follow from the above arguments. Theorem 2.23 is proved.  $\square$

*Proof of Theorem 2.22.* In the present case Lemma 2.24 also holds with  $N^0\mathcal{O}_x = \{\theta_x\} \forall x \in \mathcal{O}$ . But we need to replace the map  $\mathcal{F}$  in (2.82) by

$$\mathcal{F}(x, u^+ + u^-) : N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow \mathbb{R}, (x, u^+ + u^-) \mapsto \mathcal{L} \circ \exp_x(u^+ + u^-).$$

For any  $x \in \mathcal{O}_x$ , let  $\mathcal{F}_x$  be the restriction of  $\mathcal{F}$  to  $N^+\mathcal{O}(\epsilon)_x \oplus N^-\mathcal{O}(\epsilon)_x$ . As in the proof of Theorem 2.1, Lemma 2.25 is still true with  $\overline{N^0\mathcal{O}(2\epsilon_2)}_x = \{\theta_x\}$ . Then the desired conclusions can be obtained by applying [40, Theorem A.2] to  $\Lambda = \mathcal{O}$  and  $J_\lambda = \mathcal{F}_x$  with  $\lambda = x \in \mathcal{O}$ .  $\square$

Many results in [13, 50, 67, 68] also hold in our setting. Here are a few of them, which are needed in this paper.

**Corollary 2.27** (Shifting Theorem). *Let  $\mathbf{K}$  be any commutative ring. Then*

i) *Under the assumptions of Theorem 2.22, it holds that*

$$C_*(\mathcal{L}, \mathcal{O}; \mathbb{Z}_2) \cong C_{*-\mu_{\mathcal{O}}}(\mathcal{O}; \mathbb{Z}_2), \quad (2.89)$$

$$C_*(\mathcal{L}, \mathcal{O}; \mathbf{K}) \cong C_{*-\mu_{\mathcal{O}}}(\mathcal{O}; \theta^- \otimes \mathbf{K}), \quad (2.90)$$

where  $\theta^-$  is the orientation bundle of  $N^-\mathcal{O}$ .

ii) *Under the assumptions of Theorem 2.23, if  $\mathcal{O}$  has trivial normal bundle then for any commutative group  $\mathbf{K}$  and  $x \in \mathcal{O}$ ,*

$$C_q(\mathcal{L}, \mathcal{O}; \mathbf{K}) \cong \bigoplus_{j=0}^q C_{q-j-\mu_{\mathcal{O}}}(\mathcal{L}_x^\circ, \theta_x; \mathbf{K}) \otimes H_j(\mathcal{O}; \mathbf{K}) \quad \forall q = 0, 1, \dots. \quad (2.91)$$

(Consequently, every  $C_q(\mathcal{L}, \mathcal{O}; \mathbf{K})$  is isomorphic to finite direct sum  $r_1\mathbf{K} \oplus \dots \oplus r_s\mathbf{K} \oplus H_j(\mathcal{O}; \mathbf{K})$ , where each  $r_i \in \{0, 1\}$ , see [43, Remark 4.6]. )

As in [6, 69], (2.89)–(2.90) follow from (2.76).

**Corollary 2.28.** *Under the assumptions of Corollary 2.27, we have:*

(i)  *$\mathcal{O}$  is a local minimum (so  $\mu_{\mathcal{O}} = 0$ ) if and only if*

$$C_q(\mathcal{L}, \mathcal{O}; \mathbf{K}) \cong \delta_{q0}\mathbf{K} \quad \forall q \in \mathbb{Z} \iff C_0(\mathcal{L}, \mathcal{O}; \mathbf{K}) \neq 0.$$

(ii)  *$C_1(\mathcal{L}, \mathcal{O}; \mathbf{K}) \neq 0$  and  $\text{rank} N^0\mathcal{O} = 1$  then  $\mu_{\mathcal{O}} = 0$  and*

$$C_q(\mathcal{L}, \mathcal{O}; \mathbf{K}) \cong \mathbf{K} \otimes H_{q-1}(\mathcal{O}; \mathbf{K}) \quad \forall q \in \mathbb{Z}. \quad (2.92)$$

(iii) *If  $\text{rank} N^0\mathcal{O} = 1$  in the case  $\mu_{\mathcal{O}} = 0$ , then  $\theta$  is of mountain pass type (in the sense that some (and hence any)  $\theta_x$  with  $x \in \mathcal{O}$  is a critical point of  $\mathcal{L} \circ \exp_x$  on  $N\mathcal{O}_x$  of mountain pass type) if and only if (2.92) holds;*

(iv) *If  $C_k(\mathcal{L}, \mathcal{O}; \mathbf{K}) \neq 0$  for  $k = \text{rank} N^-\mathcal{O}$  then for any  $q \in \mathbb{Z}$*

$$C_q(\mathcal{L}, \mathcal{O}; \mathbf{K}) \cong \begin{cases} \mathbf{K} \otimes \mathbf{K} & \text{if } q \geq k, \\ 0 & \text{if } q < k. \end{cases}$$



By the Peter-Weyl theorem, the compact Lie group has a faithful representation into the real orthogonal group  $O(m)$  (i.e., a injective Lie group homomorphism into  $O(m)$ ) for some integer  $m > 0$ . Hence  $G$  can be viewed as a subgroup of  $O(m)$ . Let  $E$  be the Hilbert manifold consisting of all  $m$ -orthogonal frames in the Hilbert space  $l^2$ . It is a contractible space on which  $G$  acts freely. Let  $B_G = E/G$  denote the classifying space, which is a Hilbert manifold. Then  $E \rightarrow B_G$  is a universal smooth principal  $G$ -bundle. Let  $E \times_G \mathcal{H}$  be the quotient of  $E \times \mathcal{H}$  by the free diagonal action  $g \cdot (p, u) = (gp, g \cdot u)$ . This Hilbert manifold is a fiber space on  $B_G$  with fiber  $\mathcal{H}$ .

The  $G$ -invariant functional  $\mathcal{L}$  lifts a natural one on  $E \times \mathcal{H}$ ,  $(p, u) \mapsto \mathcal{L}(u)$ , and hence induces a functional  $\mathcal{L}^E$  on  $E \times_G \mathcal{H}$  with same smoothness as  $\mathcal{L}$ . The critical orbit  $\mathcal{O}$  of  $\mathcal{L}$  corresponds to a critical manifold  $E \times_G \mathcal{O}$  of  $\mathcal{L}^E$ , and they have the same Morse indexes. Moreover, for a  $G$ -invariant Gromoll-Meyer pair  $(W, W^-)$  of  $\mathcal{O}$  (see [67] for its existence),  $(E \times_G W, E \times_G W^-)$  is a Gromoll-Meyer of  $E \times_G \mathcal{O}$ .

Let  $c = \mathcal{L}|_{\mathcal{O}}$  and  $U$  be a  $G$ -invariant neighborhood of  $\mathcal{O}$  with  $K(\mathcal{L}) \cap \mathcal{L}_c \cap U = \mathcal{O}$ , where  $\mathcal{L}_c = \{x \in \mathcal{H} \mid \mathcal{L}(x) \leq c\}$ . For any coefficient ring  $\mathbf{K}$  and  $q \in \mathbb{N} \cup \{0\}$ , the  $q^{\text{th}}$   $G$  critical group of  $\mathcal{O}$  is defined by

$$\begin{aligned} C_G^*(\mathcal{L}, \mathcal{O}; \mathbf{K}) &= H_G^*(\mathcal{L}_c \cap U, (\mathcal{L}_c \setminus \mathcal{O}) \cap U; \mathbf{K}) \\ &= H^*(E \times_G (\mathcal{L}_c \cap U), E \times_G ((\mathcal{L}_c \setminus \mathcal{O}) \cap U); \mathbf{K}). \end{aligned}$$

It is equal to  $H_G^*(W, W^-; \mathbf{K})$ , see [13, page 76]. Moreover, if  $\mathcal{O}$  is nondegenerate, it follows from (2.76) and the Thom isomorphism theorem (with twisted coefficients) that

$$C_G^*(\mathcal{L}, \mathcal{O}; \mathbf{K}) \cong H_G^{\mu_{\mathcal{O}} - 1}(\mathcal{O}; \theta^- \otimes \mathbf{K}), \quad (2.93)$$

where  $\mu_{\mathcal{O}}$  is the Morse index of  $\mathcal{O}$  and  $\theta^-$  is the orientation bundle of  $N^-\mathcal{O}$ , see Theorem 7.5 on the page 75 of [13].

More generally, the corresponding versions of Theorems 2.14, 2.18 and 2.19 can also be proved. We only write the following since it is needed in the proof of Theorem 3.20 later.

**Theorem 2.29** (Parameterized Splitting Theorem around Critical Orbits). *Under the assumptions of Theorem 2.23, suppose further that  $G$ -invariant functionals  $\mathcal{G}_j \in C^1(\mathcal{H}, \mathbb{R})$ ,  $j = 1, \dots, n$ , have value zero and vanishing derivative at each point of  $\mathcal{O}$ , and also satisfy:*

- (i) *gradients  $\nabla \mathcal{G}_j$  have Gâteaux derivatives  $\mathcal{G}_j''(u)$  at each point  $u$  near  $\mathcal{O}$ ,*
- (ii)  *$\mathcal{G}_j''(u)$  are continuous at each point  $u \in \mathcal{O}$ .*

*If the critical orbit  $\mathcal{O}$  is degenerate, i.e.,  $\text{rank} N^0 \mathcal{O} > 0$ , then for sufficiently small  $\epsilon > 0$ ,  $\delta > 0$ ,*

**(I)** *there exists a unique continuous map*

$$\mathfrak{h} : [-\delta, \delta]^n \times N^0 \mathcal{O}(3\epsilon) \rightarrow N^+ \mathcal{O} \oplus N^- \mathcal{O}, \quad (\vec{\lambda}, x, v) \mapsto \mathfrak{h}_x(\vec{\lambda}, v), \quad (2.94)$$

such that  $\mathfrak{h}(\vec{\lambda}, \cdot) : N^0\mathcal{O}(3\epsilon) \rightarrow N^+\mathcal{O} \oplus N^-\mathcal{O}$ ,  $(x, v) \mapsto \mathfrak{h}_x(\vec{\lambda}, v)$  is a  $G$ -equivariant topological bundle morphism that preserves the zero section and satisfies

$$(P_x^+ + P_x^-) \circ \Pi_x \nabla(\mathcal{L}_{\vec{\lambda}} \circ \exp_x)(v + \mathfrak{h}_x(\vec{\lambda}, v)) = 0 \quad \forall (x, v^0) \in N^0\mathcal{O}(\epsilon), \quad (2.95)$$

where  $\mathcal{L}_{\vec{\lambda}} = \mathcal{L} + \sum_{j=1}^n \mathcal{G}_j$ ;

(II) there exists a continuous map

$$\Phi : [-\delta, \delta]^n \times N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow N\mathcal{O} \quad (2.96)$$

such that  $\Phi(\vec{\lambda}, \cdot) : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow N\mathcal{O}$  is a  $G$ -equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers, and that

$$\mathcal{L}_{\vec{\lambda}} \circ \exp \circ \Phi(\vec{\lambda}, x, v, u^+ + u^-) = \|u^+\|_x^2 - \|u^-\|_x^2 + \mathcal{L}_{\vec{\lambda}} \circ \exp_x(v + \mathfrak{h}_x(\vec{\lambda}, v)) \quad (2.97)$$

for any  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $x \in \mathcal{O}$  and  $(v, u^+, u^-) \in N^0\mathcal{O}(\epsilon)_x \times N^+\mathcal{O}(\epsilon)_x \times N^-\mathcal{O}(\epsilon)_x$ ;

(III) for each  $(\vec{\lambda}, x) \in [-\delta, \delta]^n \times \mathcal{O}$  the functional

$$N^0\mathcal{O}(\epsilon)_x \rightarrow \mathbb{R}, \quad v \mapsto \mathcal{L}_{\vec{\lambda}, x}^\circ(v) := \mathcal{L}_{\vec{\lambda}} \circ \exp_x(v + \mathfrak{h}_x(\vec{\lambda}, v)) \quad (2.98)$$

is  $G_x$ -invariant, of class  $C^1$ , and satisfies

$$D\mathcal{L}_{\vec{\lambda}, x}^\circ(v)v' := (\nabla(\mathcal{L}_{\vec{\lambda}} \circ \exp_x)(v + \mathfrak{h}_x(\vec{\lambda}, v)), v'), \quad \forall v' \in N^0\mathcal{O}_x.$$

Moreover, if the critical orbit  $\mathcal{O}$  is nondegenerate, i.e.,  $\text{rank} N^0\mathcal{O} = 0$ , then  $\mathfrak{h}$  does not appear,  $\Phi$  is from  $[-\delta, \delta]^n \times N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon)$  to  $N\mathcal{O}$ , and (2.97) becomes

$$\mathcal{L}_{\vec{\lambda}} \circ \exp \circ \Phi(\vec{\lambda}, x, u^+ + u^-) = \|u^+\|_x^2 - \|u^-\|_x^2 + \mathcal{L}_{\vec{\lambda}}|_{\mathcal{O}} \quad (2.99)$$

for any  $\vec{\lambda} \in [-\delta, \delta]^n$ ,  $x \in \mathcal{O}$  and  $(u^+, u^-) \in N^+\mathcal{O}(\epsilon)_x \times N^-\mathcal{O}(\epsilon)_x$ .

## 2.7 Proof of Theorem 2.6

Without loss of generality we may assume  $\theta \in V$  and  $u_0 = \theta$ . By the assumption we have a  $C^2$  reduction functional  $\mathcal{L}^\circ : B_H(\theta, \delta) \cap H^0 \rightarrow \mathbb{R}$  such that  $\theta$  is the unique critical point of it and  $\mathcal{L}^\circ(z) = o(\|z\|^2)$ . Clearly, we can shrink  $\delta > 0$  so that  $\delta < \min\{r, 1\}$ ,  $\bar{B}_H(\theta, \delta) \cap H^0 \subset V$  and  $\omega$  in Lemma 2.8 satisfies

$$\omega(z + \varphi(z)) < \frac{1}{2} \min\{a_0, a_1\}, \quad \forall z \in B_H(\theta, \delta) \cap H^0. \quad (2.1)$$

By the uniqueness of solutions we can also require that if  $v \in B_H(\theta, \delta)$  satisfies  $(I - P^0)\nabla\mathcal{L}(v) = 0$  then  $v = z + \varphi(z)$  for some  $z \in B_H(\theta, \delta) \cap H^0$ .

Take a smooth function  $\rho : [0, \infty) \rightarrow \mathbb{R}$  satisfying:  $0 \leq \rho \leq 1$ ,  $\rho(t) = 1$  for  $t \leq \delta/2$ ,  $\rho(t) = 0$  for  $t \geq \delta$ , and  $|\rho'(t)| < 4/\delta$ . For  $b \in H^0$  we set  $\mathcal{L}_b^\circ(z) = \mathcal{L}^\circ(z) + \rho(\|z\|)(b, z)_H$ . Then

$$\begin{aligned} D\mathcal{L}_b^\circ(z)\xi &= D\mathcal{L}(z + \varphi(z))(\xi + \varphi'(z)\xi) + \rho(\|z\|)(b, \xi)_H \\ &\quad + \rho'(\|z\|)(b, z)_H(z/\|z\|, \xi)_H, \quad \forall \xi \in H^0. \end{aligned} \quad (2.2)$$

Note that  $\|D\mathcal{L}^\circ(z)\| \geq \nu$  for some  $\nu > 0$  and for all  $z \in \bar{B}_{H^0}(\theta, \delta) \setminus B_{H^0}(\theta, \delta/2)$ . Suppose  $\|b\| < \nu/5$ . Then

$$\|D\mathcal{L}_b^\circ(z)\| = \|D\mathcal{L}^\circ(z) + \rho(\|z\|)b + (b, z)_H \rho'(\|z\|)z/\|z\|\| \geq \nu - 5\|b\| > 0, \quad (2.3)$$

and therefore  $\mathcal{L}_b^\circ$  has no critical point in  $\bar{B}_{H^0}(\theta, \delta) \setminus B_{H^0}(\theta, \delta/2)$ . By Sard's theorem we may take arbitrary small  $b$  such that the critical points of  $\mathcal{L}_b^\circ$ , if any, are nondegenerate. Choose a  $C^2$  function  $\beta : H \rightarrow \mathbb{R}$  such that  $\beta(u) = 0$  for  $u \in H \setminus B_H(\theta, r)$ , and  $\beta(u) = 1$  for  $u \in B_H(\theta, \delta)$ . Clearly, we can require  $\|\beta^{(i)}(u)\| \leq M$  for some  $M > 0$ ,  $i = 0, 1, 2$  and for all  $u \in H$ . Define

$$\tilde{\mathcal{L}}_b(u) = \mathcal{L}(u) + \beta(u)\rho(\|P^0 u\|)(b, P^0 u)_H \quad (2.4)$$

Clearly, it satisfies (iii). If  $b$  is taken to be very small, then (ii) is satisfied. Moreover, since  $\mathcal{L}$  satisfies the (PS) condition,  $\|D\mathcal{L}(u)\| \geq c$  for some  $c > 0$  and for all  $u \in B_H(\theta, r) \setminus B_H(\theta, \delta)$ . Hence all critical points of  $\tilde{\mathcal{L}}_b$  belong to  $B_H(\theta, \delta)$  as long as  $b$  is small enough.

Let  $v \in B_H(\theta, \delta)$  be a critical point of  $\tilde{\mathcal{L}}_b$ . Then

$$\begin{aligned} 0 = \tilde{\mathcal{L}}'_b(v)\xi &= (\nabla \mathcal{L}(v), \xi)_H + \rho(\|P^0 v\|)(b, P^0 \xi)_H \\ &\quad + \rho'(\|P^0 v\|)(b, P^0 v)_H (P^0 v, P^0 \xi)_H / \|P^0 v\|, \quad \forall \xi \in H. \end{aligned} \quad (2.5)$$

Since  $\rho(\|P^0 v\|) = 1$  for  $\|P^0 v\| \leq \|v\| < \delta$ , this implies  $(\nabla \mathcal{L}(v), \xi)_H = 0$  for any  $\xi \in H^+ \oplus H^-$ , i.e.,  $((I - P^0)\nabla \mathcal{L}(v), \xi)_H = 0$  for any  $\xi \in H$ . It follows that  $v = z + \varphi(z)$  for some  $z \in B_H(\theta, \delta) \cap H^0$ . [This  $z$  is nonzero. Otherwise,  $v = \theta$ . But  $\theta$  is not a critical point of  $\tilde{\mathcal{L}}_b$  if  $b \neq \theta$ ]. Note that  $(\nabla \mathcal{L}(z + \varphi(z)), \varphi'(z)\xi)_H = 0 \forall \xi \in H^0$  because  $\varphi'(z)\xi \in H^+ \oplus H^-$ . (2.5) leads to

$$\begin{aligned} 0 &= (\nabla \mathcal{L}(z + \varphi(z)), \xi)_H + \rho(\|z\|)(b, \xi)_H + \rho'(\|z\|)(b, z)_H (z, \xi)_H / \|z\| \\ &= (\nabla \mathcal{L}(z + \varphi(z)), \xi)_H + (\nabla \mathcal{L}(z + \varphi(z)), \varphi'(z)\xi)_H \\ &\quad + \rho(\|z\|)(b, \xi)_H + \rho'(\|z\|)(b, z)_H (z, \xi)_H / \|z\| \quad \forall \xi \in H^0, \end{aligned}$$

and therefore  $D\mathcal{L}_b^\circ(z) = 0$  by (2.2). That is,  $z$  is a critical point of  $\mathcal{L}_b^\circ$ , and so  $z \in B_{H^0}(\theta, \delta/2)$  by (2.3). It follows from (2.4) that

$$\tilde{\mathcal{L}}''_b(v)(\xi, \eta) = (\mathcal{L}''(v)\xi, \eta)_H, \quad \forall \xi, \eta \in H. \quad (2.6)$$

Let us prove that  $v$  is a nondegenerate critical point of  $\tilde{\mathcal{L}}_b$ . Suppose  $\xi \in \text{Ker}(\tilde{\mathcal{L}}''_b(v))$ . Then  $(\tilde{\mathcal{L}}''_b(v)\xi, \eta)_H = 0$  for any  $\eta \in H$ . By (2.6), we have

$$\tilde{\mathcal{L}}''(v)(\xi, \eta) = (\mathcal{L}''(z + \varphi(z))\xi, \eta)_H = 0, \quad \forall \eta \in H. \quad (2.7)$$

Decompose  $\xi$  into  $\xi^0 + \xi^\perp$ , where  $\xi^0 \in H^0$  and  $\xi^\perp \in H^+ \oplus H^-$ . A direct computation yields

$$(\mathcal{L}''(z + \varphi(z))\xi^0, \eta + \varphi'(z)\eta)_H + (\mathcal{L}''(z + \varphi(z))\xi^\perp, \eta + \varphi'(z)\eta)_H = 0, \quad \forall \eta \in H^0. \quad (2.8)$$

Note that  $(I - P^0)\nabla\mathcal{L}(w + \varphi(w)) = 0$  for any  $w \in B_{H^0}(\theta, \delta)$ . Hence  $(\nabla\mathcal{L}(w + \varphi(w)), \zeta)_H = 0$  for any  $w \in B_{H^0}(\theta, \delta)$  and  $\zeta \in H^+ \oplus H^-$ . Differentiating this equality with respect to  $w$  yields

$$(\mathcal{L}''(w + \varphi(w))(\tau + \varphi'(w)\tau), \zeta)_H = 0, \quad \forall \tau \in H^0, \forall w \in B_{H^0}(\theta, \delta), \forall \zeta \in H^+ \oplus H^-.$$

In particular, we have  $(\mathcal{L}''(z + \varphi(z))\xi^\perp, \eta + \varphi'(z)\eta)_H = 0$  for all  $\eta \in H^0$ . This and (2.8) yield

$$d^2\mathcal{L}^\circ(z)(\xi^0, \eta) = (\mathcal{L}''(z + \varphi(z))\xi^0, \eta + \varphi'(z)\eta)_H = 0, \quad \forall \eta \in H^0. \quad (2.9)$$

Moreover,  $d^2\mathcal{L}_b^\circ(z') = d^2\mathcal{L}^\circ(z')$  for any  $z' \in B_{H^0}(\theta, \delta/2)$  by the construction of  $\mathcal{L}_b^\circ$ . We obtain that  $d^2\mathcal{L}_b^\circ(z)(\xi, \eta) = 0$  for all  $\eta \in H^0$ . Hence  $\xi^0 = \theta$  since  $z$  is a nondegenerate critical point of  $\mathcal{L}_b^\circ$  by the choice of  $b$ , and thus  $\xi = \xi^\perp$ . By (2.6)–(2.7) and  $(\tilde{\mathcal{L}}_b''(v)\xi, \eta)_H = 0 \quad \forall \eta \in H$ , we get

$$(\mathcal{L}''(z + \varphi(z))\xi^\perp, \eta)_H = (\mathcal{L}''(z + \varphi(z))\xi, \eta)_H = 0, \quad \forall \eta \in H. \quad (2.10)$$

Hence  $\mathcal{L}''(z + \varphi(z))\xi^\perp = 0$ . Decompose  $\xi^\perp$  into  $\xi^+ + \xi^-$ , where  $\xi^+ \in H^+$  and  $\xi^- \in H^-$ . Then  $\mathcal{L}''(z + \varphi(z))\xi^+ = -\mathcal{L}''(z + \varphi(z))\xi^-$ . By Lemma 2.8 and (2.1) we derive

$$\begin{aligned} a_1\|\xi^+\|^2 &\leq (\mathcal{L}''(z + \varphi(z))\xi^+, \xi^+)_H = -(\mathcal{L}''(z + \varphi(z))\xi^-, \xi^+)_H \leq \frac{a_1}{2}\|\xi^+\| \cdot \|\xi^-\|, \\ -a_0\|\xi^-\|^2 &\geq (\mathcal{L}''(z + \varphi(z))\xi^-, \xi^-)_H = -(\mathcal{L}''(z + \varphi(z))\xi^+, \xi^-)_H \geq -\frac{a_0}{2}\|\xi^-\| \cdot \|\xi^+\|. \end{aligned}$$

These imply  $\|\xi^+\| \leq \|\xi^-\|/2$  and  $\|\xi^-\| \leq \|\xi^+\|/2$ . Hence  $\xi^+ = \xi^- = \theta$  and so  $\xi^\perp = \theta$ . This shows that  $v$  is a nondegenerate critical point of  $\tilde{\mathcal{L}}_b$ . Lemma 2.8 and (2.6) give rise to

$$\begin{aligned} \tilde{\mathcal{L}}_b''(v)(\xi, \xi) &= (\mathcal{L}''(v)\xi, \xi)_H \geq a_1\|\xi\|^2, \quad \forall \xi \in H^+, \\ \tilde{\mathcal{L}}_b''(v)(\xi, \xi) &= (\mathcal{L}''(v)\xi, \xi)_H \leq -a_0\|\xi\|^2, \quad \forall \xi \in H^-. \end{aligned}$$

But  $H = H^+ \oplus H^0 \oplus H^-$ ,  $\dim H^- = m^-$  and  $\dim H^0 = n^0$ . These show that the Morse index of  $\tilde{\mathcal{L}}_b''(v)$  must sit in  $[m^-, m^- + n^0]$ . (iv) is proved.

We also need to show that  $\tilde{\mathcal{L}}_b$  satisfies Hypothesis 2.5 on  $V$  if  $b$  is small enough. By (2.4) we have for all  $\xi, \eta \in H$ ,

$$\begin{aligned} \tilde{\mathcal{L}}_b'(u)\xi &= \mathcal{L}'(u)\xi + (\beta'(u)\xi)\rho(\|P^0u\|)(b, P^0u)_H + \beta(u)\rho(\|P^0u\|)(b, P^0\xi)_H \\ &\quad + \beta(u)\rho'(\|P^0u\|)(b, P^0u)_H(P^0u, P^0\xi)_H/\|P^0u\| \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} (\tilde{\mathcal{L}}_b''(u)\eta, \xi)_H &= (\mathcal{L}''(u)\eta, \xi)_H + (\beta''(u)\eta, \xi)_H\rho(\|P^0u\|)(b, P^0u)_H \\ &\quad + (\beta'(u)\xi)\rho(\|P^0u\|)(b, P^0\eta)_H + (\beta'(u)\xi)(b, P^0u)_H\rho'(\|P^0u\|)(P^0u, P^0\eta)_H/\|P^0u\| \\ &\quad + (\beta'(u)\eta)\rho(\|P^0u\|)(b, P^0\xi)_H + \beta(u)(b, P^0\xi)_H\rho'(\|P^0u\|)(P^0u, P^0\eta)_H/\|P^0u\| \\ &\quad + (\beta'(u)\eta)\rho'(\|P^0u\|)(b, P^0u)_H(P^0u, P^0\xi)_H/\|P^0u\| \\ &\quad + \beta(u)(\rho''(\|P^0u\|)(P^0u, P^0\eta)_H/\|P^0u\|)(b, P^0u)_H(P^0u, P^0\xi)_H/\|P^0u\| \\ &\quad + \beta(u)\rho'(\|P^0u\|)(b, P^0\eta)_H(P^0u, P^0\xi)_H/\|P^0u\| \\ &\quad + \beta(u)\rho'(\|P^0u\|)(b, P^0u)_H[(P^0\eta, P^0\xi)_H/\|P^0u\| - (P^0u, P^0\xi)_H(P^0u, P^0\eta)_H/\|P^0u\|^3] \\ &= (\mathcal{L}''(u)\eta, \xi)_H + \Upsilon(u, b, \xi, \eta). \end{aligned}$$

By the constructions of  $\beta$  and  $\rho$  we have a constant  $M_2 > 0$  such that

$$|\Upsilon(u, b, \xi, \eta)| \leq M_2 \|b\| \cdot \|\xi\| \cdot \|\eta\|$$

for all  $u \in V$  and  $\xi, \eta \in H$ . Since we may require that the support of  $\beta$  can be contained a neighborhood of  $\theta$  on which (iii) of Hypothesis 2.5 holds, for sufficiently small  $b$  the positive definite part of  $\tilde{\mathcal{L}}'_b$ ,  $\tilde{P}$  given by  $(\tilde{P}(u)\xi, \eta)_H = (P(u)\xi, \eta)_H + \Upsilon(u, b, \xi, \eta)$ , is also uniformly positive definite on this neighborhood. Hence  $\tilde{\mathcal{L}}_b$  satisfies Hypothesis 2.5.

By (2.4) and (2.11) we have positive numbers  $M_i, i = 0, 1$ , such that  $|\tilde{\mathcal{L}}_b(u) - \mathcal{L}(u)| \leq M_0 \|b\|$  and  $\|\tilde{\mathcal{L}}'_b(u)\xi - \mathcal{L}'(u)\xi\| \leq M_1 \|b\| \cdot \|\xi\|$  for all  $u \in V$  and  $\xi \in H$ . So (ii) and (iii) can be satisfied if  $b$  is small.

Finally, let us prove that  $\tilde{\mathcal{L}}_b$  satisfies the (PS) condition for small  $b$ . By (ii) and (iii) in Hypothesis 2.5, there exists  $\epsilon \in (0, \delta/2)$  such that for all  $u \in B_H(\theta, \epsilon)$  and  $\xi \in H$ ,

$$(P(u)\xi, \xi)_H \geq C_0 \|\xi\|^2 \quad \text{and} \quad \|Q(u) - Q(\theta)\| < C_0/2. \quad (2.12)$$

Since  $\mathcal{L}$  satisfies the (PS) condition and  $\theta$  is a unique critical point of  $\mathcal{L}$  in  $V$ , we have  $\nu_0 > 0$  such that  $\|\mathcal{L}'(u)\| \geq \nu_0$  for all  $u \in V \setminus B_H(\theta, \epsilon)$ . Let us choose  $b$  so small that  $\|\tilde{\mathcal{L}}'_b(u)\| \geq \nu_0/2$  for all  $u \in V \setminus B_H(\theta, \epsilon)$ . Then if  $\{u_n\}_n \subset V$  satisfies  $\tilde{\mathcal{L}}'_b(u_n) \rightarrow 0$  and  $\sup_n |\tilde{\mathcal{L}}_b(u_n)| < \infty$ , then  $\{u_n\}_n$  must be contained in  $B_H(\theta, \epsilon)$ . It follows that  $\nabla \tilde{\mathcal{L}}_b(u_n) = \nabla \mathcal{L}(u_n) + b$  for all  $n$ . For any two natural numbers  $n$  and  $m$ , using the mean value theorem we have  $\tau \in (0, 1)$  such that

$$\begin{aligned} (\nabla \mathcal{L}(u_n) - \nabla \mathcal{L}(u_m), u_n - u_m)_H &= (B(\tau u_n + (1 - \tau)u_m)(u_n - u_m), u_n - u_m)_H \\ &= (P(\tau u_n + (1 - \tau)u_m)(u_n - u_m), u_n - u_m)_H + (Q(\theta)(u_n - u_m), u_n - u_m)_H \\ &\quad + ([Q(\tau u_n + (1 - \tau)u_m) - Q(\theta)](u_n - u_m), u_n - u_m)_H \\ &\geq C_0 \|u_n - u_m\|^2 - \frac{C_0}{2} \|u_n - u_m\|^2 + (Q(\theta)(u_n - u_m), u_n - u_m)_H \end{aligned}$$

by (2.12). Passing to a subsequence we may assume  $u_n \rightharpoonup u_0$ . Since  $Q(\theta)$  is compact,  $Q(\theta)u_n \rightarrow Q(\theta)u_0$  and so  $(Q(\theta)(u_n - u_m), u_n - u_m)_H \rightarrow 0$  as  $n, m \rightarrow \infty$ . Note that  $\nabla \mathcal{L}(u_n) - \nabla \mathcal{L}(u_m) = (\nabla \mathcal{L}(u_n) + b) - (\nabla \mathcal{L}(u_m) + b) \rightarrow 0$  as  $n, m \rightarrow \infty$ . It follows from the above inequalities that  $\|u_n - u_m\| \rightarrow 0$  as  $n, m \rightarrow \infty$ . This implies  $u_n \rightarrow u_0$ . Theorem 2.6 is proved.

### 3 Bifurcations for potential operators

In this section, some previous bifurcation theorems, such as those by Rabinowitz [57], by Fadell and Rabinowitz [27, 28], and by Chang and Wang [15, 67, 68], by Chow and Lauterbach [18] and by Bartsch and Clapp [1, 2], were generalized so that they can be used to study variational bifurcation for the functional  $\mathcal{F}$  in (1.8). Our methods are mainly based on our Morse lemma, Theorem 2.1, and the parameterized splitting and shifting theorems, Theorems 2.18, 2.19. The latter suggest that multiparameter bifurcations can be studied similarly; we here give two, Theorems 3.3, 3.4.

### 3.1 Generalizations of a bifurcation theorem by Chow and Lauterbach

Let  $H$  be a real Hilbert space,  $I$  an open interval containing 0 in  $\mathbb{R}$ , and  $\{B_\lambda\}_{\lambda \in I}$  a family of bounded linear self-adjoint operators on  $H$  such that  $\|B_\lambda - B_0\| \rightarrow 0$  as  $\lambda \rightarrow 0$ . Suppose that 0 is an isolated point of the spectrum  $\sigma(B_0)$  with  $n = \dim \text{Ker}(B_0) \in (0, \infty)$ , and that  $\text{Ker}(B_\lambda) = \{0\} \forall \pm \lambda \in (0, \varepsilon_0)$  for some positive number  $\varepsilon_0 \ll 1$ . By the arguments on the pages 107 and 203 in [33], for each  $\lambda \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ ,  $B_\lambda$  has  $n$  eigenvalues near zero, and none of them is zero. In Kato's terminology in [33, page 107], we have the so-called 0-group  $\text{eig}_0(B_\lambda)$  consisting of eigenvalues of  $B_\lambda$  which approach 0 as  $\lambda \rightarrow 0$ . Let  $r(B_\lambda)$  be the number of elements in  $\text{eig}_0(B_\lambda) \cap \mathbb{R}^-$  and

$$r_{B_\lambda}^+ = \lim_{\lambda \rightarrow 0^+} r(B_\lambda), \quad r_{B_\lambda}^- = \lim_{\lambda \rightarrow 0^-} r(B_\lambda). \quad (3.1)$$

**Theorem 3.1.** *Let  $U$  be an open neighborhood of the origin of a real Hilbert space  $H$ , and  $I$  an open interval containing 0 in  $\mathbb{R}$ ,  $\mathcal{F} \in C^0(I \times V, \mathbb{R})$  such that  $\mathcal{L} := \mathcal{F}_\lambda = \mathcal{F}(\lambda, \cdot)$  satisfy Hypothesis 1.1 with  $X = H$  for each  $\lambda \in I$ . Suppose that one of the following two conditions is satisfied.*

- (1) *For some small  $\delta > 0$ ,  $\lambda \mapsto \mathcal{F}_\lambda$  is continuous at  $\lambda = 0$  in  $C^1(\bar{B}_H(\theta, \delta))$  topology.*
- (2) *For some small  $\delta > 0$ ,  $\lambda \mapsto \mathcal{F}_\lambda$  is continuous at  $\lambda = 0$  in  $C^0(\bar{B}_H(\theta, \delta))$  topology; and for every sequences  $\lambda_n \rightarrow \lambda_0$  in  $I$  and  $\{u_n\}_{n \geq 1} \subset \bar{B}_H(\theta, \delta)$  with  $\mathcal{F}'_{\lambda_n}(u_n) \rightarrow \theta$  and  $\{\mathcal{F}_{\lambda_n}(u_n)\}_{n \geq 1}$  bounded, there exists a subsequence  $u_{n_k} \rightarrow u_0 \in \bar{B}_H(\theta, \delta)$  with  $\mathcal{F}'_{\lambda_0}(u_0) = 0$ .*

Then

- (I) *If  $(\theta, 0)$  is not a bifurcation point of the equation*

$$\mathcal{F}'_u(\lambda, u) = 0, \quad (\lambda, u) \in I \times V, \quad (3.2)$$

*(i.e.,  $(0, \theta)$  is not in the closure of  $\{(\lambda, u) \in I \times V \mid \mathcal{F}'_u(\lambda, u) = 0, u \neq \theta\}$ ), then critical groups  $C_*(\mathcal{F}_\lambda, \theta; \mathbf{K})$  are well-defined and have no changes as  $\lambda$  varies in a small neighborhood of 0.*

- (II)  *$(0, \theta)$  is a bifurcation point of the equation (3.2) if the following conditions are also satisfied:*

- (a)  $\text{Ker}(d^2 \mathcal{F}_\lambda(\theta)) = \{\theta\}$  for small  $|\lambda| \neq 0$ ;
- (b)  $d^2 \mathcal{F}_\lambda(\theta) \rightarrow d^2 \mathcal{F}_0(\theta)$  as  $\lambda \rightarrow 0$ ;
- (c)  $0 \in \sigma(d^2 \mathcal{F}_0(\theta))$  (and so is an isolated point of the spectrum  $\sigma(d^2 \mathcal{F}_0(\theta))$ ) and an eigenvalue of  $d^2 \mathcal{F}_0(\theta)$  of the finite multiplicity by [5, Lemma 2.2]);
- (d)  $r_{d^2 \mathcal{F}_\lambda(\theta)}^+ \neq r_{d^2 \mathcal{F}_\lambda(\theta)}^-$ .

*Proof. Step 1.* This is a direct consequence of the stability of critical groups. In fact, since  $(\theta, 0)$  is not a bifurcation point of the equation (3.2), we may find  $0 < \varepsilon_0 \ll 1$  and a small bounded neighborhood  $W$  of  $\theta \in H$  with  $\overline{W} \subset B_H(\theta, \delta)$  such that for each  $\lambda \in (-\varepsilon_0, \varepsilon_0)$  the functional  $\mathcal{F}_\lambda$  has a unique critical point  $\theta$  sitting in  $\overline{W}$ . Note that  $\mathcal{F}_\lambda$  is of class  $(S)_+$ . We can assume that it satisfies the (PS) condition in  $\overline{W}$  by shrinking  $W$  (if necessary). After shrinking  $\varepsilon_0 > 0$  (if necessary), we may use the stability of critical groups (cf. [16, Theorem III.4] and [22, Theorem 5.1]) to derive

$$C_*(\mathcal{F}_\lambda, \theta; \mathbf{K}) = C_*(\mathcal{F}_0, \theta; \mathbf{K}), \quad \forall \lambda \in (-\varepsilon_0, \varepsilon_0) \quad (3.3)$$

provided that (1) holds. If (2) is satisfied the same claim is obtained by [19, Theorem 3.6].

**Step 2.** By a contradiction, suppose that  $(\theta, 0)$  is not a bifurcation point of the equation (3.2). Then we have (3.3) from (I). By (a),  $\theta$  is a nondegenerate critical point of  $\mathcal{F}_\lambda$ . It follows from (3.3) and Theorem 2.1 that all  $\mathcal{F}_\lambda$ ,  $0 < |\lambda| < \varepsilon_0$ , have the same Morse index  $\mu_\lambda$  at  $\theta \in H$ , i.e.,  $(-\varepsilon_0, \varepsilon_0) \setminus \{0\} \ni \lambda \mapsto \mu_\lambda$  is constant.

By [40, Proposition B.2], each  $\varrho \in \sigma(d^2\mathcal{F}_0(\theta)) \cap \{t \in \mathbb{R}^- \mid t \leq 0\}$  is an isolated point in  $\sigma(d^2\mathcal{F}_0(\theta))$ , which is also an eigenvalue of finite multiplicity. (This can also be derived from [5, Lemma 2.2]). Since  $0 \in \sigma(d^2\mathcal{F}_0(\theta))$  by (c), we may assume  $\sigma(d^2\mathcal{F}_0(\theta)) \cap \{t \in \mathbb{R}^- \mid t \leq 0\} = \{0, \varrho_1, \dots, \varrho_k\}$ , where  $\mu_i$  has multiplicity  $s_i$  for each  $i = 1, \dots, k$ . As above, by this, (b) and the arguments on the pages 107 and 203 in [33], if  $0 < |\lambda|$  is small enough,  $d^2\mathcal{F}_\lambda(\theta)$  has exactly  $s_i$  (possible same) eigenvalues near  $\mu_i$ , but total dimension of corresponding eigensubspaces is equal to that of eigensubspace of  $\varrho_i$ . Hence if  $\lambda \in (0, \varepsilon_0)$  (resp.  $-\lambda \in (0, \varepsilon_0)$ ) is small enough we obtain

$$\mu_\lambda = \mu_0 + r_{d^2\mathcal{F}_\lambda(\theta)}^+ \quad (\text{resp. } \mu_{-\lambda} = \mu_0 + r_{d^2\mathcal{F}_\lambda(\theta)}^-).$$

These and (d) imply  $\mu_\lambda - \mu_{-\lambda} = r_{d^2\mathcal{F}_\lambda(\theta)}^+ - r_{d^2\mathcal{F}_\lambda(\theta)}^- \neq 0$  for small  $\lambda \in (0, \varepsilon_0)$ , which contradicts the above claim that  $(-\varepsilon_0, \varepsilon_0) \setminus \{0\} \ni \lambda \mapsto \mu_\lambda$  is constant.  $\square$

Part (II) in Theorem 3.1 is a partial generalization of a bifurcation theorem due to [18]. The latter requires: 1)  $\mathcal{F} \in C^2(I \times V, \mathbb{R})$  (so (b) holds naturally), 2)  $0 < \dim \text{Ker}(d^2\mathcal{F}_0(\theta)) < \infty$ , 3) 0 is isolated in  $\sigma(d^2\mathcal{F}_0(\theta))$ , 4) (d) is satisfied. Its proof is based on center manifold theory, which is different from ours.

### 3.2 Generalizations of Rabinowitz bifurcation theorem

Since the birth of the Rabinowitz bifurcation theorem [57] some generalizations and new proofs are given, see [15, 17], [22], [32] and [67, 68], etc.

Our generalization will reduce to the following result, which may be obtained as a corollary of [32, Theorem 2].

**Theorem 3.2** ([11, Theorem 5.1]). *Let  $X$  be a finite dimensional normed space, let  $\delta > 0$ ,  $\lambda^* \in \mathbb{R}$  and for every  $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$ , let  $\phi_\lambda : B(\theta, \delta) \rightarrow \mathbb{R}$  be a function of class  $C^1$ . Assume that*

- a)** the functions  $\{(\lambda, u) \rightarrow \phi_\lambda(u)\}$  and  $\{(\lambda, u) \rightarrow \phi'_\lambda(u)\}$  are continuous on  $[\lambda^* - \delta, \lambda^* + \delta] \times B(\theta, \delta)$ ;
- b)**  $u = \theta$  is a critical point of  $\phi_{\lambda^*}$ ;  $\phi_\lambda$  has an isolated local minimum (maximum) at zero for every  $\lambda \in (\lambda^*, \lambda^* + \delta]$  and an isolated local maximum (minimum) at zero for every  $\lambda \in [\lambda^* - \delta, \lambda^*)$ .

Then one at least of the following assertions holds:

- i)**  $u = \theta$  is not an isolated critical point of  $\phi_{\lambda^*}$ ;
- ii)** for every  $\lambda \neq \lambda^*$  in a neighborhood of  $\lambda^*$  there is a nontrivial critical point of  $\phi_\lambda$  converging to zero as  $\lambda \rightarrow \lambda^*$ ;
- iii)** there is a one-sided (right or left) neighborhood of  $\lambda^*$  such that for every  $\lambda \neq \lambda^*$  in the neighborhood there are two distinct nontrivial critical points of  $\phi_\lambda$  converging to zero as  $\lambda \rightarrow \lambda^*$ .

It was generalized to infinite dimension spaces in [22, Theorem 4.2].

The following is a generalization of the necessity part of Theorem 12 in [58, Chapter 4, §4.3] (including the classical Krasnoselski potential bifurcation theorem [37]). The sufficiency part of Theorem 12 in [58, Chapter 4, §4.3] is contained in the case that the condition (a) in Theorem 3.5 holds.

**Theorem 3.3.** Let  $U$  be an open neighborhood of the origin of a real Hilbert space  $H$ . Suppose

- (i)**  $\mathcal{F} \in C^1(U, \mathbb{R})$  satisfies Hypothesis 1.1 with  $X = H$  as the functional  $\mathcal{L}$  there;
- (ii)**  $\mathcal{G}_j \in C^1(U, \mathbb{R})$  satisfies  $\mathcal{G}'_j(\theta) = \theta$ ,  $j = 1, \dots, n$ , and each gradient  $\mathcal{G}'_j$  has the Gâteaux derivative  $\mathcal{G}''_j(u)$  at any  $u \in U$ , which is compact linear operators and satisfy (D3) in Hypothesis 1.1 with  $X = H$ .

If  $(\vec{\lambda}^*, \theta) \in \mathbb{R}^n \times U$  is a (multiparameter) bifurcation point for the equation

$$\mathcal{F}'(u) = \sum_{j=1}^n \lambda_j \mathcal{G}'_j(u), \quad u \in U, \quad (3.4)$$

then  $\vec{\lambda}^* = (\lambda_1^*, \dots, \lambda_n^*)$  is an eigenvalue of

$$\mathcal{F}''(\theta)v - \sum_{j=1}^n \lambda_j \mathcal{G}''_j(\theta)v = 0, \quad v \in H, \quad (3.5)$$

in other words,  $\theta$  is a degenerate critical point of the functional  $\mathcal{F} - \sum_{j=1}^n \lambda_j^* \mathcal{G}_j$ . (The solution space of (3.5), denoted by  $H(\vec{\lambda})$ , is of finite dimension because it is the kernel of a Fredholm operator).



Theorem 12 in [58, Chapter 4, §4.3] also required: (a)  $\mathcal{G}$  is weakly continuous and uniformly differentiable in  $U$ , (b)  $\mathcal{F}'$  has uniformly positive definite Frechét derivatives and satisfies the condition  $\alpha$ ) in [58, Chapter 3, §2.2]. If  $\mathcal{G}'$  is completely continuous (i.e., mapping a weakly convergent sequence into a convergent one in norm) and has Frechét derivative  $\mathcal{G}''(u)$  at  $u \in U$ , then  $\mathcal{G}''(u) \in \mathcal{L}(H)$  is a compact linear operator (cf. [4, Remark 2.4.6]).

*Proof of Theorem 3.3.* Let  $(\vec{\lambda}^*, \theta) \in \mathbb{R}^n \times U$  be a bifurcation point for (3.4). Then we have a sequence  $(\vec{\lambda}_k, u_k) \in \mathbb{R}^n \times (U \setminus \{\theta\})$  such that  $\vec{\lambda}_k = (\lambda_{k,1}, \dots, \lambda_{k,n}) \rightarrow \vec{\lambda}^*$ ,  $u_k \rightarrow \theta$  and

$$\mathcal{F}'(u_k) = \sum_{j=1}^n \lambda_{k,j} \mathcal{G}'_j(u_k), \quad k = 1, 2, \dots.$$

Passing to a subsequence, if necessary, we can assume  $v_k = u_k / \|u_k\| \rightharpoonup v^*$ . By the assumptions,  $B = \mathcal{F}''$  has a decomposition  $P + Q$  as in Hypothesis 1.1 with  $X = H$ . (D4) and Lemma 2.9 imply that there exist positive constants  $\eta_0 > 0$  and  $C'_0 > 0$  such that

$$(P(u)h, h) \geq C'_0 \|h\|^2 \quad \forall h \in H, \quad \forall u \in B_H(\theta, \eta_0) \subset U. \quad (3.6)$$

Clearly, we can assume that  $\{u_k\}_{k \geq 1}$  is contained in  $B_H(\theta, \eta_0)$ . Note that

$$\frac{1}{\|u_k\|^2} (\mathcal{F}'(u_k), u_k) = \sum_{j=1}^n \frac{\lambda_{k,j}}{\|u_k\|^2} (\mathcal{G}'_j(u_k), u_k), \quad k = 1, 2, \dots. \quad (3.7)$$

Since  $\mathcal{G}'_j(\theta) = \theta$ ,  $j = 1, \dots, n$ , using the Mean Value Theorem we have a sequence  $\{t_k\}_{k \geq 1} \subset (0, 1)$  such that

$$\begin{aligned} \sum_{j=1}^n \frac{\lambda_{k,j}}{\|u_k\|^2} (\mathcal{G}'_j(u_k), u_k) &= \sum_{j=1}^n \lambda_{k,j} (\mathcal{G}''_j(t_k u_k) v_k, v_k) = \sum_{j=1}^n \lambda_{k,j} ([\mathcal{G}''_j(t_k u_k) - \mathcal{G}''_j(\theta)] v_k, v_k) \\ &+ \sum_{j=1}^n \lambda_{k,j} (\mathcal{G}''_j(\theta) v_k, v_k) \rightarrow \sum_{j=1}^n \lambda_j^* (\mathcal{G}''_j(\theta) v^*, v^*) \end{aligned} \quad (3.8)$$

because all  $\mathcal{G}''_j(\theta)$  are compact and  $\|\mathcal{G}''_j(t_k u_k) - \mathcal{G}''_j(\theta)\| \rightarrow 0$  by (D3). Moreover, since  $\mathcal{F}'(\theta) = \theta$ , we may use the Mean Value Theorem to yield a sequence  $\{s_k\}_{k \geq 1} \subset (0, 1)$  such that

$$\begin{aligned} \frac{1}{\|u_k\|^2} (\mathcal{F}'(u_k), u_k) &= \frac{1}{\|u_k\|^2} (\mathcal{F}''(s_k u_k) u_k, u_k) \\ &= \frac{1}{\|u_k\|^2} (P(s_k u_k) u_k, u_k) + \frac{1}{\|u_k\|^2} (Q(s_k u_k) u_k, u_k) \\ &\geq C'_0 + \frac{1}{\|u_k\|^2} (Q(s_k u_k) u_k, u_k) \quad \forall k \in \mathbb{N} \end{aligned}$$

by (3.6). As in (3.8) we have also

$$\frac{1}{\|u_k\|^2} (Q(s_k u_k) u_k, u_k) \rightarrow (Q(\theta) v^*, v^*).$$

It follows from these and (3.7) that  $C'_0 \leq ([\sum_{j=1}^n \lambda_j^* \mathcal{G}''_j(\theta) - Q(\theta)] v^*, v^*)$  and hence  $v^* \neq \theta$ .

Moreover, for any  $h \in H$  we have

$$\frac{1}{\|u_k\|}(\mathcal{F}'(u_k), h) = \sum_{k=1}^n \frac{\lambda_k}{\|u_k\|}(\mathcal{G}'_j(u_k), h), \quad k = 1, 2, \dots, \quad (3.9)$$

and as in (3.8) we may prove that

$$\sum_{j=1}^n \frac{\lambda_k}{\|u_k\|}(\mathcal{G}'_j(u_k), h) \rightarrow \sum_{j=1}^n \lambda_j^*(\mathcal{G}''_j(\theta)v^*, h), \quad (3.10)$$

and that for some sequence  $\{\tau_k\}_{k \geq 1} \subset (0, 1)$ , depending on  $\{u_k\}_{k \geq 1}$  and  $h$ ,

$$\frac{1}{\|u_k\|}(\mathcal{F}'(u_k), h) = (\mathcal{F}''(\tau_k u_k)v_k, h) = (v_k, \mathcal{F}''(\tau_k u_k)h) \rightarrow (v^*, \mathcal{F}''(\theta)h)$$

because  $v_k \rightharpoonup v^*$  and  $\|\mathcal{F}''(\tau_k u_k)h - \mathcal{F}''(\theta)h\| \rightarrow 0$  by (D2) and (D3). This and (3.9)–(3.10) lead to

$$\sum_{j=1}^n \lambda_j^*(\mathcal{G}''_j(\theta)v^*, h) = (v^*, \mathcal{F}''(\theta)h) \quad \forall h \in H$$

and hence  $\mathcal{F}''(\theta)v^* - \sum_{j=1}^n \lambda_j^* \mathcal{G}''_j(\theta)v^* = 0$ . That is,  $\vec{\lambda}^*$  is an eigenvalue of (3.15).  $\square$

In the following we give two generalizations of Rabinowitz bifurcation theorem [57].

**Theorem 3.4.** *Under the assumptions (i)–(ii) of Theorem 3.3, suppose:*

- (iii)  $\vec{\lambda}^*$  is an isolated eigenvalue of (3.5);
- (iv) the corresponding finite dimension reduction  $\mathcal{L}_{\vec{\lambda}}^\circ$  with the functional  $\mathcal{L} = \mathcal{F} - \sum_{j=1}^n \lambda_j^* \mathcal{G}_j$  as in Theorem 2.18 is of class  $C^2$  for each  $\vec{\lambda}$  near the origin of  $\mathbb{R}^n$ ;
- (v) (2.62) holds with  $H^0 = H(\vec{\lambda}^*)$  (the solution space of (3.5) with  $\vec{\lambda} = \vec{\lambda}^*$ );
- (vi) either  $\dim H(\vec{\lambda}^*)$  is odd or there exists  $\vec{\lambda} \in \mathbb{R}^n \setminus \{\vec{0}\}$  such that the symmetric bilinear form

$$H(\vec{\lambda}^*) \times H(\vec{\lambda}^*) \ni (z_1, z_2) \mapsto \mathcal{Q}_{\vec{\lambda}}(z_1, z_2) = \sum_{j=1}^n \lambda_j(\mathcal{G}''_j(\theta)z_1, z_2)_H \quad (3.11)$$

has different Morse index and coindex. (In particular, if  $n = 1$  the latter is equivalent to the fact that the form  $H(\lambda^*) \times H(\lambda^*) \ni (z_1, z_2) \mapsto (\mathcal{G}''_1(\theta)z_1, z_2)_H$  has different Morse index and co-index).

Then  $(\vec{\lambda}^*, \theta) \in \mathbb{R}^n \times U$  is a bifurcation point for the equation (3.4). Furthermore, if for some  $\vec{\mu} \in \mathbb{R}^n \setminus \{\vec{0}\}$  the form  $\mathcal{Q}_{\vec{\mu}}$  defined by (3.11) is either positive definite or negative one, then one of the following alternatives occurs:

- (A)  $(\vec{\lambda}^*, \theta)$  is not an isolated solution of (3.4) in  $\{\vec{\lambda}^*\} \times U$ .
- (B) there exists a sequence  $\{t_k\}_{k \geq 1} \subset \mathbb{R} \setminus \{0\}$  such that  $t_k \rightarrow 0$  and that for each  $t_k$  the equation (3.4) with  $\vec{\lambda} = t_k \vec{\mu} + \vec{\lambda}^*$  has infinitely many solutions converging to  $\theta \in H$ .

(C) for every  $t$  in a small neighborhood of  $0 \in \mathbb{R}$  there is a nontrivial solution  $u_t$  of (3.4) with  $\vec{\lambda} = t\vec{\mu} + \vec{\lambda}^*$  converging to  $\theta$  as  $t \rightarrow 0$ ;

(D) there is a one-sided  $\mathfrak{T}$  neighborhood of  $0 \in \mathbb{R}$  such that for any  $t \in \mathfrak{T} \setminus \{0\}$ , (3.4) with  $\vec{\lambda} = t\vec{\mu} + \vec{\lambda}^*$  has at least two nontrivial solutions converging to zero as  $t \rightarrow 0$ .

*Proof. Step 1.* By the assumptions we have the conclusions of Theorem 2.18 with  $\mathcal{L} = \mathcal{F} - \sum_{j=1}^n \lambda_j^* \mathcal{G}_j$ . Suppose that  $(\vec{\lambda}^*, \theta) \in \mathbb{R}^n \times U$  is not a bifurcation point for the equation (3.4). Then as in the proof of Theorem 3.1(I) we may find  $0 < \eta \ll 1$  with  $\bar{B}_H(\theta, \eta) \subset U$  such that after shrinking  $\delta > 0$  in Theorem 2.18 for each  $\vec{\lambda} \in [-\delta, \delta]^n$  the following claims hold:

- the functional  $\mathcal{L}_{\vec{\lambda}}$  has a unique critical point  $\theta$  in  $\bar{B}_H(\theta, \eta)$ ,
- $\vec{\lambda}^* = (\lambda_1^*, \dots, \lambda_n^*)$  is a unique eigenvalue of (3.5) in  $[-\delta, \delta]^n + \vec{\lambda}^*$ ,
- for all  $\vec{\lambda} \in [-\delta, \delta]^n$ ,

$$C_*(\mathcal{L}_{\vec{\lambda}}, \theta; \mathbf{K}) = C_*(\mathcal{L}_{\vec{0}}, \theta; \mathbf{K}) = C_*(\mathcal{F} - \sum_{j=1}^n \lambda_j^* \mathcal{G}_j, \theta; \mathbf{K}). \quad (3.12)$$

We may also shrink  $\epsilon > 0, r > 0, s > 0$  and  $W$  in Theorem 2.18 so that

$$\bar{B}_{H^0}(\theta, \epsilon) \oplus \bar{B}_{H^+}(\theta, r) \oplus \bar{B}_{H^-}(\theta, s) \subset B_H(\theta, \eta) \quad \text{and} \quad \bar{W} \subset B_H(\theta, \eta),$$

where  $H^0 = H(\vec{\lambda}^*)$ . By (3.12) and Theorem 2.19 we have

$$C_*(\mathcal{L}_{\vec{\lambda}}^\circ, \theta; \mathbf{K}) = C_*(\mathcal{L}_{\vec{0}}^\circ, \theta; \mathbf{K}), \quad \forall \vec{\lambda} \in [-\delta, \delta]^n. \quad (3.13)$$

(This can also be derived from the stability of critical groups as before.) For each  $\vec{\lambda} \in [-\delta, \delta]^n \setminus \{\vec{0}\}$ , since  $\theta \in H$  is a nondegenerate critical point of  $\mathcal{L}_{\vec{\lambda}}$ , Claim below (2.62) tells us that  $\theta \in H^0$  is a nondegenerate critical point of  $\mathcal{L}_{\vec{\lambda}}^\circ$  too. Hence (3.13) implies that the Morse index of  $\mathcal{L}_{\vec{\lambda}}^\circ$  at  $\theta$  is constant with respect to  $\vec{\lambda} \in [-\delta, \delta]^n \setminus \{\vec{0}\}$ .

On the other hand, by (vi), if  $\dim H(\vec{\lambda}^*)$  is odd, for every  $\vec{\lambda} \in [-\delta, \delta]^n \setminus \{\vec{0}\}$  the nondegenerate quadratic forms  $d^2 \mathcal{L}_{\vec{\lambda}}^\circ(\theta)$  and  $d^2 \mathcal{L}_{-\vec{\lambda}}^\circ(\theta)$  on  $H(\vec{\lambda}^*)$  must have different Morse indexes, where  $d^2 \mathcal{L}_{\vec{\lambda}}^\circ(\theta) : H(\vec{\lambda}^*) \times H(\vec{\lambda}^*) \rightarrow \mathbb{R}$  given by

$$d^2 \mathcal{L}_{\vec{\lambda}}^\circ(\theta)(z_1, z_2) = (\mathcal{L}_{\vec{\lambda}}''(\theta)z_1, z_2)_H = - \sum_{j=1}^n \lambda_j (\mathcal{G}_j''(\theta)z_1, z_2)_H = -\mathcal{Q}_{\vec{\lambda}}(z_1, z_2).$$

This contradicts (3.13). Similarly, if there exists  $\vec{\lambda} \in \mathbb{R}^n \setminus \{\vec{0}\}$  such that the form in (3.11) has different Morse index and coindex, then for every  $t > 0$  with  $t\vec{\lambda} \in [-\delta, \delta]^n$  the forms  $d^2 \mathcal{L}_{t\vec{\lambda}}^\circ(\theta)$  and  $d^2 \mathcal{L}_{-t\vec{\lambda}}^\circ(\theta)$  on  $H(\vec{\lambda}^*)$  have different Morse indexes, and hence a contradiction is obtained.

**Step 2.** By replacing  $\vec{\mu}$  by  $-\vec{\mu}$  we may assume that the form  $\mathcal{Q}_{\vec{\mu}}$  is positive definite. Suppose that any one of (A)–(C) does not hold. Then there exists  $\epsilon \in (0, 1)$  such that  $\theta \in H(\vec{\lambda}^*)$  is an isolated critical point of  $\mathcal{L}_{t\vec{\mu}}^\circ$  for each  $t \in [-\epsilon, \epsilon]$ . By the assumption  $d^2 \mathcal{L}_{t\vec{\mu}}^\circ(\theta)$  is negative (resp. positive) definite for each  $t$  in  $(0, \epsilon]$  (resp.  $[-\epsilon, 0)$ ). Then Theorem 3.2 implies that for some one-sided  $\mathfrak{T}$

neighborhood of  $0 \in [-\epsilon, \epsilon]$  and any  $t \in \mathfrak{T} \setminus \{0\}$  the functional  $\mathcal{L}_{t\vec{\mu}}^\circ$  has two distinct nontrivial critical points  $z_{t,1}$  and  $z_{t,2}$  converging to  $\theta \in H(\vec{\lambda}^*)$ . Then  $u_{t,j} = z_{t,j} + \psi(t\vec{\mu} + \vec{\lambda}^*, z_{t,j})$ ,  $j = 1, 2$ , are two nontrivial solutions of (3.4) with  $\vec{\lambda} = t\vec{\mu} + \vec{\lambda}^*$ , and both converge to zero as  $t \rightarrow 0$ .  $\square$

Note that (3.5) has no isolated eigenvalues if  $\cap_{j=1}^n \text{Ker}(\mathcal{G}_j''(\theta)) \cap \text{Ker}(\mathcal{F}''(\theta)) \neq \{\theta\}$ . It is natural to ask when  $\vec{\lambda}^*$  is an isolated eigenvalue of (3.5). For the sake of simplicity let us consider the case  $n = 1$ . Then  $H(\vec{\lambda}^*) = \text{Ker}(\mathcal{F}''(\theta)) - \lambda_1^* \mathcal{G}_1''(\theta)$ , and if  $\lambda_1^* \neq 0$  we have

$$d^2 \mathcal{L}_{\vec{\lambda}^*}^\circ(\theta)(z_1, z_2) = -\lambda_1(\mathcal{G}_1''(\theta)z_1, z_2)_H = -\frac{\lambda_1}{\lambda_1^*}(\mathcal{F}''(\theta)z_1, z_2).$$

In Theorem 3.4, the final condition that the form in (3.11) is either positive definite or negative one suggests that we should require  $\mathcal{F}''(\theta)$  to be nondegenerate on  $H(\vec{\lambda}^*)$ .

Suppose now that  $\mathcal{F}''(\theta)$  is invertible,  $n = 1$  and write  $\mathcal{G} = \mathcal{G}_1$ . Then (3.14) and (3.15) become

$$\mathcal{F}'(u) = \lambda \mathcal{G}'(u), \quad u \in U, \quad (3.14)$$

$$\mathcal{F}''(\theta)v - \lambda \mathcal{G}''(\theta)v = 0, \quad v \in H, \quad (3.15)$$

respectively. Moreover, 0 is not an eigenvalue of (3.15), and  $\lambda \in \mathbb{R} \setminus \{0\}$  is an eigenvalue of (3.15) if and only if  $1/\lambda$  is an eigenvalue of compact linear self-adjoint operator  $L := [\mathcal{F}''(\theta)]^{-1} \mathcal{G}''(\theta) \in \mathcal{L}_s(H)$ . By Riesz-Schauder theory, the spectrum of  $L$ ,  $\sigma(L)$ , contains a unique accumulation point 0, and  $\sigma(L) \setminus \{0\}$  is a real countable set of eigenvalues of finite multiplicity, denoted by  $\{1/\lambda_n\}_{n=1}^\infty$ . Let  $H_n$  be the eigensubspace corresponding to  $1/\lambda_n$  for  $n \in \mathbb{N}$ . Then  $H = \oplus_{n=0}^\infty H_n$ ,  $H_0 = \text{Ker}(L) = \text{Ker}(\mathcal{G}''(\theta))$  and

$$H_n = \text{Ker}(I/\lambda_n - L) = \text{Ker}(\mathcal{F}''(\theta) - \lambda_n \mathcal{G}''(\theta)), \quad n = 1, 2, \dots \quad (3.16)$$

As another generalization of Rabinowitz bifurcation theorem [57] we have the following improvement of sufficiency of Theorem 12 in [58, Chap.4, §4.3].

**Theorem 3.5.** *Let  $\mathcal{F}, \mathcal{G} = \mathcal{G}_1 \in C^1(U, \mathbb{R})$  be as in Theorem 3.3, and  $\lambda^*$  be an eigenvalue of (3.15). Suppose that the operator  $\mathcal{F}''(\theta)$  is invertible and also satisfies one of the following three conditions: (a) positive definite, (b) negative definite, (c) each  $H_n$  in (3.16) with  $L = [\mathcal{F}''(\theta)]^{-1} \mathcal{G}''(\theta)$  is an invariant subspace of  $\mathcal{F}''(\theta)$  (e.g. these are true if  $\mathcal{F}''(\theta)$  commutes with  $\mathcal{G}''(\theta)$ ), and  $\mathcal{F}''(\theta)$  is either positive definite or negative one on  $H_{n_0}$  if  $\lambda^* = \lambda_{n_0}$ . Then  $(\lambda^*, \theta) \in \mathbb{R} \times U$  is a bifurcation point for the equation (3.14) and one of the following alternatives occurs:*

- (i)  $(\lambda^*, \theta)$  is not an isolated solution of (3.14) in  $\{\lambda^*\} \times U$ .
- (ii) there exists a sequence  $\{\kappa_n\}_{n \geq 1} \subset \mathbb{R} \setminus \{\lambda^*\}$  such that  $\kappa_n \rightarrow \lambda^*$  and that for each  $\kappa_n$  the equation (3.14) with  $\lambda = \kappa_n$  has infinitely many solutions converging to  $\theta \in H$ .
- (iii) for every  $\lambda$  in a small neighborhood of  $\lambda^*$  there is a nontrivial solution  $u_\lambda$  of (3.14) converging to  $\theta$  as  $\lambda \rightarrow \lambda^*$ ;

(iv) *there is a one-sided  $\Lambda$  neighborhood of  $\lambda^*$  such that for any  $\lambda \in \Lambda \setminus \{\lambda^*\}$ , (3.14) has at least two nontrivial solutions converging to zero as  $\lambda \rightarrow \lambda^*$ .*

It is easily seen that the functional  $\mathcal{F}$  in [58, §4.3, Theorem 4.3] or in [59, Chap.1, Theorem 3.4] satisfies the conditions of this theorem for the case (a).

*Proof of Theorem 3.5. Case 1.  $\mathcal{F}''(\theta)$  is either positive definite or negative one.*

Clearly, we only need to consider the first case. For  $\lambda > \lambda_n$  and  $h \in H_n \setminus \{\theta\}$  with  $n > 0$ , since  $\lambda_n \mathcal{G}''(\theta)h = \mathcal{F}''(\theta)h$ , we have

$$(\mathcal{F}''(\theta)h - \lambda \mathcal{G}''(\theta)h, h) = (1 - \lambda/\lambda_n)(\mathcal{F}''(\theta)h, h) < 0. \quad (3.17)$$

Clearly, if  $H_0 \neq \{\theta\}$  (this is true if  $\dim H = \infty$ ), for  $h \in H_0 \setminus \{\theta\}$  it holds that

$$(\mathcal{F}''(\theta)h - \lambda \mathcal{G}''(\theta)h, h) = (\mathcal{F}''(\theta)h, h) > 0.$$

Let  $\mu_\lambda$  denote the Morse index of  $\mathcal{L}_\lambda := \mathcal{F} - \lambda \mathcal{G}$  at  $\theta$ . Then by (3.17) we obtain

$$\mu_\lambda = \sum_{\lambda_n < \lambda} \dim H_n. \quad (3.18)$$

Assume  $\lambda^* = \lambda_{n_0}$  for some  $n_0 \in \mathbb{N}$ . Then we have  $\varepsilon > 0$  such that  $(\lambda^* - 2\varepsilon, \lambda^* + 2\varepsilon) \setminus \{\lambda^*\}$  has no intersection with  $\{\lambda_n\}_{n=1}^\infty$  since  $\lambda_n \rightarrow \infty$ . By (3.18) it is easy to verify that

$$\mu_\lambda = \mu_{\lambda^*}, \quad \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*], \quad (3.19)$$

$$\mu_\lambda = \mu_{\lambda^*} + \nu_{\lambda^*}, \quad \forall \lambda \in (\lambda^*, \lambda^* + 2\varepsilon), \quad (3.20)$$

where  $\nu_{\lambda^*} = \dim H_{n_0}$  is the nullity of  $\mathcal{L}_{\lambda^*}$  at  $\theta$ .

By Step 1 of proof of Theorem 2.14, we have

**Claim 1.** *After shrinking  $\varepsilon > 0$  we may verify that the homotopy*

$$[\lambda^* - \varepsilon, \lambda^* + \varepsilon] \times \overline{B_H(\theta, \varepsilon)} \rightarrow H, \quad (\lambda, v) \mapsto \nabla \mathcal{L}_\lambda(v)$$

*is of class  $(S)_+$ . So if  $\{(\kappa_n, v_n)\}_{n \geq 1} \subset [\lambda^* - \varepsilon, \lambda^* + \varepsilon] \times \overline{B_H(\theta, \varepsilon)}$  satisfies  $\nabla \mathcal{L}_{\kappa_n}(v_n) \rightarrow \theta$  and  $\kappa_n \rightarrow \kappa_0$ , then  $\{v_n\}_{n \geq 1}$  has a convergent subsequence in  $\overline{B_H(\theta, \varepsilon)}$ .*

If (i) or (ii) holds, then  $(\lambda^*, \theta)$  is a bifurcation point for (3.14).

Now suppose that neither (i) nor (ii) holds. Then we have

**Claim 2.**  *$\theta \in H$  is an isolated critical point of  $\mathcal{L}_\lambda$  for each  $\lambda \in [\lambda^* - \varepsilon, \lambda^* + \varepsilon]$  by shrinking  $\varepsilon > 0$  if necessary.*

Writing  $H^0 = H_{n_0}$  and applying Theorem 2.18 to  $\mathcal{L}_\lambda = \mathcal{L}_{\lambda^*} - (\lambda^* - \lambda)\mathcal{G}$  with  $\lambda \in [\lambda^* - \varepsilon, \lambda^* + \varepsilon]$  and  $-\mathcal{G}$ , we have  $\delta \in (0, \varepsilon]$ ,  $\epsilon > 0$  and a unique continuous map

$$\psi : [\lambda^* - \delta, \lambda^* + \delta] \times B_H(\theta, \epsilon) \cap H^0 \rightarrow (H^0)^\perp \quad (3.21)$$

such that for each  $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$ ,  $\psi(\lambda, \theta) = \theta$  and

$$P^\perp \nabla \mathcal{F}(z + \psi(\lambda, z)) - \lambda P^\perp \nabla \mathcal{G}(z + \psi(\lambda, z)) = \theta \quad \forall z \in B_H(\theta, \epsilon) \cap H^0, \quad (3.22)$$

where  $P^\perp$  is the orthogonal projection onto  $(H^0)^\perp$ , and that the functional

$$B_H(\theta, \epsilon) \cap H^0 \ni z \mapsto \mathcal{L}_\lambda^\circ(z) := \mathcal{F}(z + \psi(\lambda, z)) - \lambda \mathcal{G}(z + \psi(\lambda, z)) \quad (3.23)$$

is of class  $C^1$ , whose differential is given by

$$D\mathcal{L}_\lambda^\circ(z)h = D\mathcal{F}(z + \psi(\lambda, z))h - \lambda D\mathcal{G}(z + \psi(\lambda, z))h, \quad \forall h \in H^0. \quad (3.24)$$

Hence the problem is reduced to finding the critical points of  $\mathcal{L}_\lambda^\circ$  near  $\theta \in H^0$  for fixed  $\lambda$  near  $\lambda^*$ .

Note that Claim 2 is equivalent to the following

**Claim 3.**  $\theta \in H^0$  is an isolated critical point of  $\mathcal{L}_\lambda^\circ$  for each  $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$  by shrinking  $\delta > 0$  if necessary.

Hence if  $\theta \in H^0$  is a local maximizer (resp. minimizer) of  $\mathcal{L}_\lambda^\circ$ , it must be strict.

For a  $C^1$  function  $\varphi$  on a neighborhood  $U$  of the origin  $\theta \in \mathbb{R}^N$  we may always find  $\tilde{\varphi} \in C^1(\mathbb{R}^N, \mathbb{R})$  such that it agrees with  $\varphi$  near  $\theta \in \mathbb{R}^N$  and is also coercive (so satisfies the (PS)-condition). Suppose that  $\theta$  is an isolated critical point of  $\varphi$ . By Proposition 6.95 and Example 6.45 in [52] we have

$$\left. \begin{aligned} C_k(\varphi, \theta; \mathbf{K}) = \delta_{k0} &\iff \theta \text{ is a local minimizer of } \varphi, \\ C_k(\varphi, \theta; \mathbf{K}) = \delta_{kN} &\iff \theta \text{ is a local maximizer of } \varphi, \end{aligned} \right\} \quad (3.25)$$

and  $C_0(\varphi, \theta; \mathbf{K}) = 0 = C_N(\varphi, \theta; \mathbf{K})$  if  $\theta \in \mathbb{R}^N$  is neither a local maximizer nor a local minimizer of  $\varphi$ .

Then by Theorem 2.1, (2.69) and (3.19)–(3.20) we get that for any  $q \in \mathbb{N} \cup \{0\}$ ,

$$\begin{aligned} \delta_{q(\mu_{\lambda^*} + \nu_{\lambda^*})} \mathbf{K} &= C_q(\mathcal{L}_\lambda, \theta; \mathbf{K}) = C_{q-\mu_{\lambda^*}}(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}), \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta], \\ \delta_{q\mu_{\lambda^*}} \mathbf{K} &= C_q(\mathcal{L}_\lambda, \theta; \mathbf{K}) = C_{q-\mu_{\lambda^*}}(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}), \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*). \end{aligned}$$

It follows that

$$\begin{aligned} C_j(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) &= \delta_{(j+\mu_{\lambda^*})(\mu_{\lambda^*} + \nu_{\lambda^*})} \mathbf{K} = \delta_{j\nu_{\lambda^*}} \mathbf{K}, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta], \\ C_j(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) &= \delta_{(j+\mu_{\lambda^*})\mu_{\lambda^*}} \mathbf{K} = \delta_{j0} \mathbf{K}, \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*). \end{aligned}$$

These and (3.25) imply

$$\theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^* - \nu, \lambda^*), \quad (3.26)$$

$$\theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta]. \quad (3.27)$$

By (3.26)–(3.27) and Theorem 3.2, one of the following possibilities occurs:

- (I) for every  $\lambda$  in a small neighborhood of  $\lambda^*$ ,  $\mathcal{L}_\lambda^\circ$  has a nontrivial critical point converging to  $\theta \in H^0$  as  $\lambda \rightarrow \lambda^*$ ;
- (II) there is a one-sided  $\Lambda$  neighborhood of  $\lambda^*$  such that for any  $\lambda \in \Lambda \setminus \{\lambda^*\}$ ,  $\mathcal{L}_\lambda^\circ$  has two nontrivial critical points converging to zero as  $\lambda \rightarrow \lambda^*$ .

Obviously, they lead to (iii) and (iv), respectively.

**Case 2.** Each  $H_n$  is an invariant subspace of  $\mathcal{F}''(\theta)$ ,  $n = 1, 2, \dots$ . Note that  $H_n$  has an orthogonal decomposition  $H_n^+ \oplus H_n^-$ , where  $H_n^+$  (resp.  $H_n^-$ ) is the positive (resp. negative) definite subspace of  $\mathcal{F}''(\theta)|_{H_n}$ . It is possible that  $H_n^+ = \{\theta\}$  or  $H_n^- = \{\theta\}$ .

As in (3.17), if  $H_n^+ \neq \{\theta\}$  (resp.  $H_n^- \neq \{\theta\}$ ) and  $\lambda > \lambda_n$  (resp.  $\lambda < \lambda_n$ ) we have

$$(\mathcal{F}''(\theta)h - \lambda \mathcal{G}''(\theta)h, h) = (1 - \lambda/\lambda_n)(\mathcal{F}''(\theta)h, h) < 0$$

for  $h \in H_n^+ \neq \{\theta\}$  (resp.  $h \in H_n^- \neq \{\theta\}$ ). Then the Morse index of  $\mathcal{L}_\lambda$  at  $\theta$ ,

$$\mu_\lambda = \sum_{\lambda_n < \lambda} \dim H_n^+ + \sum_{\lambda_n > \lambda} \dim H_n^-. \quad (3.28)$$

Since  $\lambda^* = \lambda_{n_0}$ , as in (3.19)-(3.20) it follows from these that

$$\mu_\lambda = \mu_{\lambda^*} + \nu_{\lambda^*}^-, \quad \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*), \quad (3.29)$$

$$\mu_\lambda = \mu_{\lambda^*} + \nu_{\lambda^*}^+, \quad \forall \lambda \in (\lambda^*, \lambda^* + 2\varepsilon), \quad (3.30)$$

where  $\nu_{\lambda^*}^+ = \dim H_{n_0}^+$  (resp.  $\nu_{\lambda^*}^- = \dim H_{n_0}^-$ ) is the positive (resp. negative) index of inertia of  $\mathcal{F}''(\theta)|_{H_{n_0}}$ .

Since  $\mathcal{F}''(\theta)$  is either positive definite or negative one on  $H_{n_0}$ , we have either

$$\mu_\lambda = \mu_{\lambda^*}, \quad \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*), \quad (3.31)$$

$$\mu_\lambda = \mu_{\lambda^*} + \nu_{\lambda^*}, \quad \forall \lambda \in (\lambda^*, \lambda^* + 2\varepsilon), \quad (3.32)$$

or

$$\mu_\lambda = \mu_{\lambda^*} + \nu_{\lambda^*}, \quad \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*), \quad (3.33)$$

$$\mu_\lambda = \mu_{\lambda^*}. \quad \forall \lambda \in (\lambda^*, \lambda^* + 2\varepsilon), \quad (3.34)$$

We also suppose that neither (i) nor (ii) holds. Then Claim 1 and so Claim 2 holds. (3.31)–(3.32) and (3.33)–(3.34) lead, respectively, to

$$C_j(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) = \delta_{j\nu_{\lambda^*}} \mathbf{K}, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta],$$

$$C_j(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) = \delta_{j0} \mathbf{K}, \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*),$$

and

$$C_j(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) = \delta_{j\nu_{\lambda^*}} \mathbf{K}, \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*),$$

$$C_j(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) = \delta_{j0} \mathbf{K}, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta].$$

The former yields (3.26)–(3.27) as above. Similarly, the latter and (3.25) imply

$$\theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*),$$

$$\theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta].$$

They and Theorem 3.2 show that either (c) or (d) holds.  $\square$

**Remark 3.6.** Suppose that (iv) is replaced by the following weaker conclusion

(iv') there is a one-sided  $\Lambda$  neighborhood of  $\lambda^*$  such that for any  $\lambda \in \Lambda \setminus \{\lambda^*\}$ , (3.14) has a nontrivial solution  $u_\lambda$  converging to zero as  $\lambda \rightarrow \lambda^*$ .

Then the finite dimension reduction in the proof is not needed; and the assumption “ $\mathcal{F}''(\theta)$  is either positive definite or negative one on  $H_{n_0}$ ” in (c) may be replaced by the weaker condition “ $\mathcal{F}''(\theta)|_{H_{n_0}}$  has nonzero signature”.

### 3.3 Bifurcation for equivariant problems

Now let us generalize the above results to the equivariant case. The first is a partial generalization of [67, Theorem 4.2] and [68, Theorem 2.5]. The proofs of the latter were based on Morse theory ideas of [15]. Because of our theory in Section 2, the same methods may be used with some technical improvements.

**Theorem 3.7.** *Under the assumptions of Theorem 3.5, let  $G$  be a compact Lie group acting on  $H$  orthogonally, and suppose that  $U$ ,  $\mathcal{F}$  and  $\mathcal{G}$  are  $G$ -invariant. For an eigenvalue  $\lambda^*$  of (3.15), suppose that the operator  $\mathcal{F}''(\theta)$  is invertible and also satisfies one of the following three conditions: (a) positive definite, (b) negative definite, (c) each  $H_n$  in (3.16) with  $L = [\mathcal{F}''(\theta)]^{-1}\mathcal{G}''(\theta)$  is an invariant subspace of  $\mathcal{F}''(\theta)$  (e.g. these are true if  $\mathcal{F}''(\theta)$  commutes with  $\mathcal{G}''(\theta)$ ), and  $\mathcal{F}''(\theta)$  is either positive definite or negative one on  $H^0 = H_{n_0}$  if  $\lambda^* = \lambda_{n_0}$ . Then*

- 1)  $(\lambda^*, \theta) \in \mathbb{R} \times U$  is a bifurcation point for the equation (3.14).
- 2) If  $\dim H_{n_0} \geq 2$  and the unit sphere in  $H_{n_0}$  is not a  $G$ -orbit we must get one of the following alternatives:
  - (i)  $(\lambda^*, \theta)$  is not an isolated solution of (3.14) in  $\{\lambda^*\} \times U$ ;
  - (ii) there exists a sequence  $\{\kappa_n\}_{n \geq 1} \subset \mathbb{R} \setminus \{\lambda^*\}$  such that  $\kappa_n \rightarrow \lambda^*$  and that for each  $\kappa_n$  the equation (3.14) with  $\lambda = \kappa_n$  has infinitely many  $G$ -orbits of solutions converging to  $\theta \in H$ ;
  - (iii) for every  $\lambda$  in a small neighborhood of  $\lambda^*$  there is a nontrivial solution  $u_\lambda$  of (3.14) converging to  $\theta$  as  $\lambda \rightarrow \lambda^*$ ;
  - (iv) there is a one-sided  $\Lambda$  neighborhood of  $\lambda^*$  such that for any  $\lambda \in \Lambda \setminus \{\lambda^*\}$ , (3.14) has at least two nontrivial critical orbits converging to zero  $\theta$  as  $\lambda \rightarrow \lambda^*$ .
- 3) Suppose one of the following assumptions holds: 3.a)  $G = T^m$  and  $\text{Fix}(G) \cap H_{n_0} = \{\theta\}$ ; 3.b)  $G = T^m$  and every orbit in  $H_{n_0}$  is homeomorphic to some  $T^s$  for  $s \geq 2$ ; 3.c)  $G$  is a finite group and the greatest common divisor  $\delta$  of set  $\{\#G/\#G_x \mid x \in H_{n_0} \setminus \{\theta\}\}$  is equal to or bigger than 2, where  $\#S$  denotes the number of elements in a set  $S$ . Then either one of the above (i)-(iii) or the following hold:



(iv)' there is a one-sided  $\Lambda$  neighborhood of  $\lambda^*$  such that for any  $\lambda \in \Lambda \setminus \{\lambda^*\}$ , (3.14) has at least  $\dim H_{n_0}$  (resp.  $2 \dim H_{n_0}$ ,  $\delta \dim H_{n_0}$ ) nontrivial critical orbits in the case 3.a) (resp. 3.b), 3.c)), where every orbit is counted with its multiplicity. (The multiplicity of a critical orbit  $\mathcal{O}$  of  $\mathcal{L}_\lambda = \mathcal{F} - \lambda \mathcal{G}$  was defined as  $c(\mathcal{O}) = \sum_{q=0}^{\infty} \text{rank} C_q(\mathcal{L}_\lambda, \mathcal{O})$  in [68, Definition 1.3]).

This theorem can also be viewed generalizations of [27, 28]. Even so, we still give a direct generalization version of [27, 28] in Theorem 3.9 below (because different methods need be employed). Another different point is that the number of critical orbits in 3) is counted in a non-usual way. After Theorem 3.9 we shall compare these two theorems.

*Proof of Theorem 3.7.* 1) follows from Theorem 3.5. For 2), as in the proof of Theorem 3.5, we assume that neither (i) nor (ii) holds. Then  $\theta \in H^0$  is a unique critical orbit of the functional  $B_H(\theta, \epsilon) \cap H^0 \ni z \mapsto \mathcal{L}_\lambda^\circ(z)$  in (3.23) for each  $\lambda \in [\lambda^* - 2\delta, \lambda^* + 2\delta]$  by shrinking  $\delta > 0$  and  $\epsilon > 0$  if necessary. (Thus if  $\theta$  is an extreme point of  $\mathcal{L}_\lambda^\circ$ , it must be strict). Moreover, since  $\dim H^0 < \infty$ , replacing  $\epsilon$  by a slightly smaller one we can assume that  $\{\mathcal{L}_\lambda^\circ \mid \lambda \in [\lambda^* - 2\delta, \lambda^* + 2\delta]\}$  satisfies the (PS) condition. That is, if  $z_k \in B_H(\theta, \epsilon) \cap H^0$  and  $\lambda_k \in [\lambda^* - 2\delta, \lambda^* + 2\delta]$  satisfy  $D\mathcal{L}_{\lambda_k}^\circ(z_k) \rightarrow 0$  and  $\sup_k |\mathcal{L}_{\lambda_k}^\circ(z_k)| < \infty$ , then  $\{(z_k, \lambda_k)\}_{k=1}^\infty$  has a converging subsequence.

The problem is reduced to finding the critical orbits of  $\mathcal{L}_\lambda^\circ$  near  $\theta \in H^0$  for fixed  $\lambda$  near  $\lambda^*$ .

Firstly, we assume that  $\mathcal{F}''(\theta)$  is positive definite.

Since  $\theta \in H^0$  is an isolated critical orbit of  $\mathcal{L}_{\lambda^*}^\circ$ , by (2.69) and (3.25), we have three cases:

- $C_q(\mathcal{L}_{\lambda^*}, \theta; \mathbf{K}) = \delta_{q\mu_{\lambda^*}} \mathbf{K}$  if  $\theta \in H^0$  is a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$ ;
- $C_q(\mathcal{L}_{\lambda^*}, \theta; \mathbf{K}) = \delta_{q(\mu_{\lambda^*} + \nu_{\lambda^*})} \mathbf{K}$  if  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_{\lambda^*}^\circ$ ;
- $C_q(\mathcal{L}_{\lambda^*}, \theta; \mathbf{K}) = 0$  for  $q \notin (\mu_{\lambda^*}, \mu_{\lambda^*} + \nu_{\lambda^*})$  if  $\theta \in H^0$  is neither a local maximizer nor a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$ .

In the third case, the stability of critical groups implies (iii) as before.

For the first two cases, as in the proofs of (3.26)–(3.27), if  $\theta \in H^0$  is a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$  we can obtain

$$\theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^* - 2\delta, \lambda^*], \quad (3.35)$$

$$\theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in (\lambda^*, \lambda^* + 2\delta]; \quad (3.36)$$

and if  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_{\lambda^*}^\circ$  we have

$$\theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^* - 2\delta, \lambda^*), \quad (3.37)$$

$$\theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^*, \lambda^* + 2\delta]. \quad (3.38)$$

Now let us follow the proof ideas of [67, page 220] to complete the remaining arguments. But differ from the case therein our  $\mathcal{L}_\lambda^\circ$  is only of class  $C^1$ . Fortunately, the standard arguments (cf. [52, Proposition 5.57]) may yield

**Lemma 3.8.** *There exists a smooth map*

$$(B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}) \times (\lambda^* - 2\delta, \lambda^* + 2\delta) \rightarrow H, (z, \lambda) \mapsto \mathcal{V}_\lambda(z)$$

such that for each  $\lambda \in (\lambda^* - 2\delta, \lambda^* + 2\delta)$  the map  $\mathcal{V}_\lambda : B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\} \rightarrow H$  is a  $G$ -equivariant pseudo-gradient vector field for  $\mathcal{L}_\lambda^\circ$ , precisely for all  $z \in B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}$ ,

$$\|\mathcal{V}_\lambda(z)\| \leq 2\|D\mathcal{L}_\lambda^\circ(z)\| \quad \text{and} \quad \langle D\mathcal{L}_\lambda^\circ(z), \mathcal{V}_\lambda(z) \rangle \geq \frac{1}{2}\|D\mathcal{L}_\lambda^\circ(z)\|^2.$$

For completeness we are also to give the proof it, which is postponed after the proof of this theorem.

When  $\theta \in H^0$  is a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$  (and so a strict local minimizer as noted above), (3.35) and (3.36) are satisfied. Let  $c = \mathcal{L}_{\lambda^*}^\circ(\theta)$ . We can choose  $\varepsilon > 0$  so small that  $\mathcal{W}_\varepsilon = \{z \in B_H(\theta, \epsilon) \cap H^0 \mid \mathcal{L}_{\lambda^*}^\circ(z) < c + \varepsilon\}$  is a contractible neighborhood of  $\theta$ . Observe that  $\mathcal{W}_\varepsilon$  is  $G$ -invariant and the flow of  $-\mathcal{V}_{\lambda^*}$  preserves  $\mathcal{W}_\varepsilon$ . (Actually, it is not hard to construct a deformation contraction from  $\mathcal{W}_\varepsilon$  to  $\theta$  with the flow of  $-\mathcal{V}_{\lambda^*}$ ). Since  $\theta \in H^0$  is a unique critical orbit of the functional  $\mathcal{L}_\lambda^\circ$  in  $\mathcal{W}_\varepsilon$  for each  $\lambda \in [\lambda^* - 2\delta, \lambda^* + 2\delta]$ , and  $\{\mathcal{L}_\lambda^\circ \mid \lambda \in [\lambda^* - 2\delta, \lambda^* + 2\delta]\}$  satisfies the (PS) condition, by the theorem on continuous dependence of solutions of ordinary differential equations on initial values and parameters we may deduce that there exists  $0 < \bar{\delta} < 2\delta$  such that the flow of  $-\mathcal{V}_\lambda$  also preserves  $\mathcal{W}_\varepsilon$  for each  $\lambda \in [\lambda^* - \bar{\delta}, \lambda^* + \bar{\delta}]$ . For any  $\lambda \in (\lambda^*, \lambda^* + \bar{\delta}]$ , since  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_\lambda^\circ$  by (3.36),  $\mathcal{L}_\lambda^\circ$  must have a minimal critical orbit  $\mathcal{O} \neq \{\theta\}$ . Note that Theorems 3.1, 3.2 also hold in our case by Corollaries 2.27, 2.28. Repeating the other arguments in the proof of [67, page 220] we may verify that for  $\lambda \in (\lambda^*, \lambda^* + \delta]$  close to  $\lambda^*$ ,  $\mathcal{L}_\lambda^\circ$  has three critical orbits.

Similarly, if (3.37) and (3.38) hold,  $\mathcal{L}_\lambda^\circ$  has three critical orbits for each  $\lambda \in [\lambda^* - \delta, \lambda^*)$  close to  $\lambda^*$ .

By considering  $-\mathcal{F}''(\theta)$  we get the conclusion if  $\mathcal{F}''(\theta)$  is negative definite.

Next, we consider the case that the condition (c) holds.

If  $\theta \in H^0$  is a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$ , by (3.31) and (3.32) we get

$$\theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*], \quad (3.39)$$

$$\theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta]. \quad (3.40)$$

If  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_{\lambda^*}^\circ$ , by (3.33) and (3.34) we have

$$\theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in [\lambda^* - \delta, \lambda^*], \quad (3.41)$$

$$\theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad \forall \lambda \in (\lambda^*, \lambda^* + \delta]. \quad (3.42)$$

If  $\theta \in H^0$  is neither a local maximizer nor a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$ , (iii) will occur by the stability of critical groups implies as before. The proofs of the first two cases are as above.

Finally, we prove 3). As in 2) we only consider the case that  $\mathcal{F}''(\theta)$  is positive definite. The other cases can be proved similarly. We assume that any one of (i)-(iii) does not occur. Then we

have either (3.35)-(3.36) or (3.35)-(3.36). In these two cases we obtain that  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_\lambda^\circ$  for any  $\lambda \in (\lambda^*, \lambda^* + 2\delta]$  (resp. for all  $\lambda \in [\lambda^*, \lambda^* + 2\delta]$ ). Note  $\nu_{\lambda^*} = \dim H_{n_0}$ . It follows (or from Theorem 2.19) that

$$C_q(\mathcal{L}_\lambda^\circ, \theta; \mathbf{K}) = \delta_{q\nu_{\lambda^*}} \mathbf{K}, \quad q = 0, 1, \dots$$

for any  $\lambda \in (\lambda^*, \lambda^* + 2\delta]$  (resp. for all  $\lambda \in [\lambda^*, \lambda^* + 2\delta]$ ). These show that  $\theta$  is essentially the same as nondegenerate critical point of  $\mathcal{L}_\lambda^\circ$  with Morse index  $\nu_{\lambda^*} = \dim H_{n_0}$ . Obverse that the conclusions of [68, Corollary 1.3] also hold in the present case (because  $X = H_{n_0}$  has finite dimension) though  $\mathcal{L}_\lambda^\circ$  is only of class  $C^1$ . The results in the cases 3.a) and 3.b) follow as in [68]. The case 3.c) can be derived from [68, Theorem 1.3].  $\square$

*Proof of Lemma 3.8.* Note that  $\lambda \mapsto \mathcal{L}_\lambda^\circ \in C^1(B_H(\theta, \epsilon) \cap H^0)$  is continuous by (2.51), and that  $D\mathcal{L}_\lambda^\circ$  has no any zero point in  $(B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}) \times [\lambda^* - 2\delta, \lambda^* + 2\delta]$ . For any given  $z \in B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}$  and  $\lambda \in (\lambda^* - 2\delta, \lambda^* + 2\delta)$  we have an open neighborhood  $O_{(z, \lambda)}$  of  $z$  in  $B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}$ , a positive number  $r_{(z, \lambda)}$  with  $(\lambda - r_{(z, \lambda)}, \lambda + r_{(z, \lambda)}) \subset (\lambda^* - 2\delta, \lambda^* + 2\delta)$  and a unit vector  $v_{(z, \lambda)} \in H$  such that for all  $(z', \lambda') \in O_{(z, \lambda)} \times (\lambda - r_{(z, \lambda)}, \lambda + r_{(z, \lambda)})$ ,

$$\|v_{(z, \lambda)}\| \leq 2\|D\mathcal{L}_{\lambda'}^\circ(z')\| \quad \text{and} \quad \langle D\mathcal{L}_{\lambda'}^\circ(z'), v_{(z, \lambda)} \rangle \geq \frac{1}{2}\|D\mathcal{L}_{\lambda'}^\circ(z')\|^2.$$

Now all above  $O_{(z, \lambda)} \times (\lambda - r_{(z, \lambda)}, \lambda + r_{(z, \lambda)})$  form an open cover  $\mathcal{Q}$  of  $(B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}) \times (\lambda^* - 2\delta, \lambda^* + 2\delta)$ , and the latter admits a  $C^\infty$ -unit decomposition  $\{\eta_\alpha\}_{\alpha \in \Xi}$  subordinate to a locally finite refinement  $\{W_\alpha\}_{\alpha \in \Xi}$  of  $\mathcal{Q}$ . Since each  $W_\alpha$  can be contained in some open subset of form  $O_{(z, \lambda)} \times (\lambda - r_{(z, \lambda)}, \lambda + r_{(z, \lambda)})$ , we have a unit vector  $v_\alpha \in H$  such that

$$\|v_\alpha\| \leq 2\|D\mathcal{L}_{\lambda'}^\circ(z')\| \quad \text{and} \quad \langle D\mathcal{L}_{\lambda'}^\circ(z'), v_\alpha \rangle \geq \frac{1}{2}\|D\mathcal{L}_{\lambda'}^\circ(z')\|^2$$

for all  $(z', \lambda') \in W_\alpha$ . Set  $\chi = \sum_{\alpha \in \Xi} \eta_\alpha v_\alpha$ . Then it is a smooth map from  $(B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}) \times (\lambda^* - 2\delta, \lambda^* + 2\delta)$  to  $H$ , and satisfies

$$\|\chi(z, \lambda)\| \leq 2\|D\mathcal{L}_\lambda^\circ(z)\| \quad \text{and} \quad \langle D\mathcal{L}_\lambda^\circ(z), \chi(z, \lambda) \rangle \geq \frac{1}{2}\|D\mathcal{L}_\lambda^\circ(z)\|^2$$

for all  $(z, \lambda) \in (B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}) \times (\lambda^* - 2\delta, \lambda^* + 2\delta)$ . Let  $d\mu$  denote the right invariant Haar measure on  $G$ . Define

$$(B_H(\theta, \epsilon) \cap H^0 \setminus \{\theta\}) \times (\lambda^* - 2\delta, \lambda^* + 2\delta) \ni (z, \lambda) \mapsto \mathcal{V}_\lambda(z) = \int_G g^{-1} \chi(gz, \lambda) d\mu \in H.$$

It is easily checked that  $\mathcal{V}_\lambda$  satisfies requirements.  $\square$

The following is a direct generalization of Fadell–Rabinowitz theorems [27, 28].

**Theorem 3.9.** *Under the assumptions of Theorem 3.7, if the Lie group  $G$  is equal to  $\mathbb{Z}_2$  or  $S^1$ , Then  $(\lambda^*, \theta) \in \mathbb{R} \times U$  is a bifurcation point for the equation (3.14), and if  $\dim H_{n_0} \geq 2$  and the unit sphere in  $H^0 = H_{n_0}$  is not a  $G$ -orbit we must get one of the following alternatives:*

- (i)  $(\lambda^*, \theta)$  is not an isolated solution of (3.14) in  $\{\lambda^*\} \times U$ ;
- (ii) there exists a sequence  $\{\kappa_n\}_{n \geq 1} \subset \mathbb{R} \setminus \{\lambda^*\}$  such that  $\kappa_n \rightarrow \lambda^*$  and that for each  $\kappa_n$  the equation (3.14) with  $\lambda = \kappa_n$  has infinitely many  $G$ -orbits of solutions converging to  $\theta \in H$ ;
- (iii) there exist left and right neighborhoods  $\Lambda^-$  and  $\Lambda^+$  of  $\lambda^*$  in  $\mathbb{R}$  and integers  $n^+, n^- \geq 0$ , such that  $n^+ + n^- \geq \dim H^0 = \dim H_{n_0}$  and for  $\lambda \in \Lambda^- \setminus \{\lambda^*\}$  (resp.  $\lambda \in \Lambda^+ \setminus \{\lambda^*\}$ ), (3.14) has at least  $n^-$  (resp.  $n^+$ ) distinct critical  $G$ -orbits different from  $\theta$ , which converge to zero  $\theta$  as  $\lambda \rightarrow \lambda^*$ .

Different from Theorem 3.7, if  $(\lambda^*, \theta)$  is not an isolated solution of (3.14) in  $\{\lambda^*\} \times U$ , Theorem 3.9 implies that there exist sequences  $\lambda_k^+ \downarrow \lambda^*$  and  $\lambda_k^- \uparrow \lambda^*$  such that for each  $k \in \mathbb{N}$ , the numbers of non-trivial critical orbits of  $\mathcal{L}_{\lambda_k^+} = \mathcal{F} - \lambda_k^+ \mathcal{G}$  plus those of non-trivial critical orbits of  $\mathcal{L}_{\lambda_k^-}$  in  $U$  are at least  $\dim H^0$ . Moreover, these non-trivial critical orbits converge  $\theta$  as  $k \rightarrow \infty$ . Note also that the count method for critical orbits in 3) of Theorem 3.7 is not usual as the present one.

*Proof of Theorem 3.9.* We assume that the cases (i) and (ii) do not occur. Moreover, we only consider cases: **1)**  $\mathcal{F}''(\theta)$  is positive definite, **2)** (c) holds and  $\mathcal{F}''(\theta)$  is positive definite on  $H^0$ . In these cases, by the proofs of Theorems 3.5, 3.7, we obtain:

**A)** if  $\theta \in H^0$  is a local minimizer of  $\mathcal{L}_{\lambda^*}^\circ$ , i.e.,  $C_q(\mathcal{L}_{\lambda^*}^\circ, \theta; \mathbf{K}) = \delta_{q0} \mathbf{K}$ , then

$$\begin{aligned} \theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad & \forall \lambda \in [\lambda^* - \delta, \lambda^*], \\ \theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad & \forall \lambda \in (\lambda^*, \lambda^* + \delta]; \end{aligned}$$

**B)** if  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_{\lambda^*}^\circ$ , i.e.,  $C_q(\mathcal{L}_{\lambda^*}^\circ, \theta; \mathbf{K}) = \delta_{q\nu_{\lambda^*}} \mathbf{K}$ , then

$$\begin{aligned} \theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad & \forall \lambda \in [\lambda^* - \delta, \lambda^*), \\ \theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad & \forall \lambda \in [\lambda^*, \lambda^* + \delta]; \end{aligned}$$

**C)** if  $\theta \in H^0$  is neither a local maximizer of  $\mathcal{L}_{\lambda^*}^\circ$  nor a local maximizer of it, i.e.,  $C_q(\mathcal{L}_{\lambda^*}^\circ, \theta; \mathbf{K}) = 0$  for  $q = 0, \nu_{\lambda^*}$ , then

$$\begin{aligned} \theta \in H^0 \text{ is a local minimizer of } \mathcal{L}_\lambda^\circ, \quad & \forall \lambda \in [\lambda^* - \delta, \lambda^*), \\ \theta \in H^0 \text{ is a local maximizer of } \mathcal{L}_\lambda^\circ, \quad & \forall \lambda \in (\lambda^*, \lambda^* + \delta]. \end{aligned}$$

Let us shrink  $\epsilon > 0$  in (3.23) such that  $\theta$  is the only critical point of  $\mathcal{L}_\lambda^\circ$  in  $B_H(\theta, \epsilon) \cap H^0$ . Since  $\lambda \mapsto \mathcal{L}_\lambda^\circ \in C^1(B_H(\theta, \epsilon) \cap H^0)$  is continuous by (2.51), it is easy to see that

$$R_{\delta, \epsilon} := \{(\lambda, z) \in [\lambda^* - \delta, \lambda^* + \delta] \times (B_H(\theta, \epsilon) \cap H^0) \mid D\mathcal{L}_\lambda^\circ(z) \neq \theta\}$$

is an open subset in  $[\lambda^* - \delta, \lambda^* + \delta] \times (B_H(\theta, \epsilon) \cap H^0)$ , and

$$R_{\delta, \epsilon} = \{(\lambda, z) \in [\lambda^* - \delta, \lambda^* + \delta] \times B_H(\theta, \epsilon) \cap H^0 \mid z \in (B_H(\theta, \epsilon) \cap H^0) \setminus K(\mathcal{L}_\lambda^\circ)\},$$

where  $K(\mathcal{L}_\lambda^\circ)$  denotes the critical set of  $\mathcal{L}_\lambda^\circ$ . As in the proof of Lemma 3.8 we can produce a smooth map,  $R_{\delta,\epsilon} \rightarrow H^0$ ,  $(\lambda, z) \mapsto \mathcal{V}_\lambda(z)$ , such that each

$$\mathcal{V}_\lambda : B_H(\theta, \epsilon) \cap H^0 \setminus K(\mathcal{L}_\lambda^\circ) \rightarrow H^0, \quad z \mapsto \mathcal{V}_\lambda(z)$$

is a  $G$ -equivariant  $C^\infty$  pseudo-gradient vector field for  $\mathcal{L}_\lambda^\circ$ .

Replacing [57, (11.1)] (or [27, (2.4)]) for  $G = \mathbb{Z}_2$ , and [28, (8.19)] for  $G = S^1$  by

$$\frac{d\varphi_\lambda}{ds} = -\mathcal{V}_\lambda(\varphi_\lambda), \quad \varphi_\lambda(0, z) = z, \quad (3.43)$$

we can repeat the constructions in [57, §1] and [27, §8] to obtain:

**Lemma 3.10.** *There is a  $G$ -invariant open neighborhood  $\mathcal{Q}$  of  $\theta$  in  $H^0$  with compact closure  $\overline{\mathcal{Q}}$  contained in  $B_H(\theta, \epsilon) \cap H^0$  such that for every  $\lambda$  close to  $\lambda^*$ , every  $c \in \mathbb{R}$  and every  $\tau_1 > 0$ , every  $G$ -neighborhood  $U$  of  $K_{\lambda,c} := K(\mathcal{L}_\lambda^\circ) \cap \{z \in \overline{\mathcal{Q}} \mid \mathcal{L}_\lambda^\circ(z) \leq c\}$  there exists an  $\tau \in (0, \tau_1)$  and a  $G$  equivariant homotopy  $\eta : [0, 1] \times \overline{\mathcal{Q}} \rightarrow \overline{\mathcal{Q}}$  with the following properties:*

- 1°  $\eta(t, z) = z$  if  $z \in \overline{\mathcal{Q}} \setminus (\mathcal{L}_\lambda^\circ)^{-1}[c - \tau_1, c + \tau_1]$ ;
- 2°  $\eta(t, \cdot)$  is homeomorphism of  $\overline{\mathcal{Q}}$  to  $\eta(t, \overline{\mathcal{Q}})$  for each  $t \in [0, 1]$ ;
- 3°  $\eta(1, A_{\lambda, c+\tau} \setminus U) \subset A_{\lambda, c-\tau}$ , where  $A_{\lambda, d} := \{z \in \overline{\mathcal{Q}} \mid \mathcal{L}_\lambda^\circ(z) \leq d\}$ ;
- 4° if  $K_{\lambda, c} = \emptyset$ ,  $\eta(1, A_{\lambda, c+\tau}) \subset A_{\lambda, c-\tau}$ .

For  $*$  = +, −, let  $S^* = \{z \in B_H(\theta, \epsilon) \cap H^0 \mid \psi(s, z) \in B_H(\theta, \epsilon) \cap H^0, \forall s > 0\}$  and  $T^* = S^* \cap \partial \overline{\mathcal{Q}}$ . For  $G = \mathbb{Z}_2$  (resp.  $S^1$ ) let  $i_G$  denote the genus in [57] (resp. the index in [28, §7]).

**Lemma 3.11.** *Both  $T^+$  and  $T^-$  are  $G$ -invariant compact subset of  $\partial \overline{\mathcal{Q}}$ , and also satisfy*

- 1°  $\min\{\mathcal{L}_\lambda^\circ(z) \mid z \in T^+\} > 0$  and  $\max\{\mathcal{L}_\lambda^\circ(z) \mid z \in T^-\} < 0$ ;
- 2°  $i_{\mathbb{Z}_2}(T^+) + i_{\mathbb{Z}_2}(T^-) \geq \dim H^0$  and  $i_{S^1}(T^+) + i_{S^1}(T^-) \geq \frac{1}{2} \dim H^0$ .

Two inequalities in 2° are [27, Lemma 2.11] and [28, Theorem 8.30], respectively.

**Case  $G = \mathbb{Z}_2$ .** Suppose  $i_{\mathbb{Z}_2}(T^-) = k > 0$ . Let  $c_j$  be defined by [27, (2.13)], but  $\bar{Q}$  and  $g(\lambda, v)$  are replaced by  $\overline{\mathcal{Q}}$  and  $\mathcal{L}_\lambda^\circ(z)$ , respectively. We can modify the proof of (i) on the page 54 of [27] as follows:

In the above three cases **A)**, **B)** and **C)**, for each  $\lambda \in [\lambda^* - \delta, \lambda^*)$ ,  $\theta \in H^0$  is always a local (strict) minimizer of  $\mathcal{L}_\lambda^\circ$ . Therefore for arbitrary sufficiently small  $\rho > 0$ , depending on  $\lambda$ ,  $\mathcal{L}_\lambda^\circ(z) > 0$  for any  $0 < \|z\| \leq \rho$  and so

$$c_1 \geq \min_{\|z\|=\rho} \mathcal{L}_\lambda^\circ(z) > 0.$$

Other arguments are same. Hence we obtain: if  $\lambda \in [\lambda^* - \delta, \lambda^*)$  is close to  $\lambda^*$ ,  $\mathcal{L}_\lambda^\circ$  has at least  $k$  distinct pairs of nontrivial critical points, which also converge to  $\theta$  as  $\lambda \rightarrow \lambda^*$ .

Since for every  $\lambda \in (\lambda^*, \lambda^* + \delta]$ ,  $\theta \in H^0$  is a local maximizer of  $\mathcal{L}_\lambda^\circ$ , by considering  $-\mathcal{L}_\lambda^\circ$  we get: if  $i_{\mathbb{Z}_2}(T^+) = l > 0$ , for every  $\lambda \in (\lambda^*, \lambda^* + \delta]$  close to  $\lambda^*$ ,  $\mathcal{L}_\lambda^\circ$  has at least  $l$  distinct pairs of nontrivial critical points converging to  $\theta$  as  $\lambda \rightarrow \lambda^*$ .

These two claims together yield the desired result.

**Case  $G = S^1$ .** Suppose  $i_{S^1}(T^-) = k > 0$ . Similarly, for  $c_j$  defined by [28, (8.56)], we may replace [28, (8.58), (8.63)] by

$$\mathcal{L}_\lambda^\circ(x) \geq \min_{\|z\|=\rho} \mathcal{L}_\lambda^\circ(z) > 0, \quad \text{and so} \quad c_{\gamma+1} \geq \min_{\|z\|=\rho} \mathcal{L}_\lambda^\circ(z) > 0,$$

and then repeat the arguments in [28, §8] to complete the final proof. Of course, we also use the fact that  $\mathcal{L}_\lambda^\circ \rightarrow \mathcal{L}_{\lambda^*}^\circ$  uniformly on  $\overline{\mathcal{D}}$  as  $\lambda \rightarrow \lambda^*$ , which can be derived from (2.51).  $\square$

Fadell–Rabinowitz theorems in [27, 28] were also generalized to the case of arbitrary compact Lie groups for potential operators of  $C^2$  functionals by Bartsch and Clapp [2], Bartsch [1]. We now give generalizations of their some results.

Fix a set  $\mathcal{A}$  of  $G$ -spaces, a multiplicative equivariant cohomology theory  $h^*$  and an ideal  $I$  of the coefficient ring  $R = h^*(pt)$ . Recall in [1, Definition 4.1] that the  $(\mathcal{A}, h^*, I)$ -length of a  $G$ -space  $X$ ,  $(\mathcal{A}, h^*, I)$ -length( $X$ ), was defined to be the smallest integer  $k$  such that there exist  $A_1, \dots, A_k$  in  $\mathcal{A}$  with the following property: For all  $\gamma \in h^*(X)$  and for all  $\omega_i \in I \cap \ker(h^*(pt) \rightarrow h^*(A_i))$ ,  $i = 1, \dots, k$ , the product  $\omega_1 \cdot \dots \cdot \omega_k \cdot \gamma = 0$  in  $h^*(X)$ . Moreover, set  $(\mathcal{A}, h^*, I)$ -length( $X$ ) =  $\infty$  if no such  $k$  exists. Let  $SE$  denote the unit sphere in a Hilbert space  $E$ , and  $G$  be a compact Lie group acting on  $E$  orthogonally. Denote by  $\mathcal{G}$  the set of orbits occurring on  $SE$ . Let  $h^*$  be any continuous, multiplicative, equivariant cohomology theory such that  $\ker(h^*(pt) \rightarrow h^*(G/H))$  is a finitely generated ideal for all  $G/H \in \mathcal{G}$ . Taking  $I = R = h^*(pt)$  the  $(\mathcal{A}, h^*, I)$ -length of a  $G$ -space  $X$  becomes the  $(\mathcal{G}, h^*)$ -length  $\ell(X)$  defined in [2]. For a bounded closed  $G$ -neighborhood  $V$  of the origin in a  $G$ -module  $E$  it was proved in [2, Lemma 1.6] that  $\ell(\partial V) = \ell(SE)$ . If  $G = \mathbb{Z}_2$  (resp.  $S^1$ ) and  $h^* = H_G^*$ ,  $\ell$  becomes  $\text{index}_{\mathbb{R}}$  (resp.  $\text{index}_{\mathbb{C}}$ ) in [28].

**Hypothesis 3.12.** Let  $G$  be a compact Lie group acting on  $H$  orthogonally, and  $U$  a  $G$ -invariant open neighborhood of the origin of a real Hilbert space  $H$ . Let  $\mathcal{F}, \mathcal{G} = \mathcal{G}_1 \in C^1(U, \mathbb{R})$  be as in Theorem 3.3, and  $G$ -invariant. Let  $\lambda^*$  be an isolated eigenvalue of (3.15), i.e., for each  $\lambda \neq \lambda^*$  near  $\lambda^*$  the equation (3.15) has only trivial solution, and let  $H^0$  be the corresponding eigenspace. (Every eigenvalue of (3.15) is isolated if  $\mathcal{F}''(\theta)$  is invertible.) Suppose  $H^0 \cap \text{Fix}(G) = \{\theta\}$ .

As in the proof of Theorem 3.5, applying Theorem 2.18 to  $\mathcal{L}_\lambda = \mathcal{F} - \lambda\mathcal{G} = \mathcal{L}_{\lambda^*} - (\lambda^* - \lambda)\mathcal{G}$  with  $\lambda \in [\lambda^* - \varepsilon, \lambda^* + \varepsilon]$  and  $-\mathcal{G}$ , we have  $\delta \in (0, \varepsilon]$ ,  $\varepsilon > 0$  and a unique continuous map

$$\psi : [\lambda^* - \delta, \lambda^* + \delta] \times (B_H(\theta, \varepsilon) \cap H^0) \rightarrow (H^0)^\perp$$

as in (3.21), such that (3.22) and (3.23)–(3.24) hold. But the present  $\psi$  is  $G$ -equivariant and each  $\mathcal{L}_\lambda^\circ$  is  $G$ -invariant.

**Hypothesis 3.13.** Under Hypothesis 3.12, suppose that the continuous map  $\psi$  as in (3.21) is of class  $C^1$  with respect to the second variable, which implies that  $d\mathcal{L}_\lambda^\circ$  has Gâteaux derivative

$$\begin{aligned} d^2\mathcal{L}_\lambda^\circ(z)(u, v) &= (\mathcal{F}_\lambda''(z + \psi(\lambda, z))(u + D_z\psi(\lambda, z)u), v)_H \\ &\quad - \lambda(\mathcal{G}_\lambda''(z + \psi(\lambda, z))(u + D_z\psi(\lambda, z)u), v)_H \quad \forall u, v \in H^0 \end{aligned}$$

at every  $z \in B_H(\theta, \epsilon) \cap H^0$  by (2.61), and therefore for all  $u, v \in H^0$ ,

$$d^2\mathcal{L}_\lambda^\circ(\theta)(u, v) = (\lambda^* - \lambda)(\mathcal{G}''(\theta)u, v)_H = (\lambda^* - \lambda)(P^0\mathcal{G}''(\theta)|_{H^0}u, v)_H \quad (3.44)$$

because  $\psi(\lambda, \theta) = \theta$  and  $D_z\psi(\lambda, \theta) = \theta$ . Furthermore, we assume that each  $\mathcal{L}_\lambda^\circ$  is of class  $C^2$ .

Since  $\lambda^*$  is an isolated eigenvalue of (3.15), 0 is an eigenvalue of  $\mathcal{F}''(\theta) - \lambda_0\mathcal{G}''(\theta)$  with finite multiplicity, isolated in the spectrum  $\sigma(\mathcal{F}''(\theta) - \lambda_0\mathcal{G}''(\theta))$ . Hence for  $\lambda$  near  $\lambda^*$  the 0-group  $\text{eig}_0(\mathcal{F}''(\theta) - \lambda\mathcal{G}''(\theta))$  (cf. Section 3.1) is well-defined. Let  $E_\lambda$  be the generalized eigenspace of  $\mathcal{F}''(\theta) - \lambda\mathcal{G}''(\theta)$  belonging to  $\text{eig}_0(\mathcal{F}''(\theta) - \lambda\mathcal{G}''(\theta)) \cap \mathbb{R}^-$ . It is  $G$ -invariant. For  $\lambda$  and  $\lambda'$  near  $\lambda^*$  the spaces  $E_\lambda$  and  $E_{\lambda'}$  are  $G$ -isomorphic if  $(\lambda - \lambda^*)(\lambda' - \lambda^*) > 0$ . (When the latter holds the orthogonal eigenprojection (cf. [33, page 181] for the definition),  $P_\lambda : H \rightarrow E_\lambda$ , restricts to a  $G$ -isomorphism from  $E_{\lambda'}$  onto  $E_\lambda$ .) Since  $\theta \in H^0$  is a nondegenerate critical point of  $\mathcal{L}_\lambda^\circ$  for each  $\lambda \neq \lambda^*$  near  $\lambda^*$ ,  $P^0\mathcal{G}''(\theta)|_{H^0} : H^0 \rightarrow H^0$  is an isomorphism by (3.44). Let  $H_+^0$  and  $H_-^0$  be the positive and negative definite subspaces of  $P^0\mathcal{G}''(\theta)|_{H^0}$ , respectively. Then  $H^0 = H_+^0 \oplus H_-^0$ . Let  $F_\lambda^-$  (resp.  $F_\lambda^+$ ) be the eigenspace belonging to  $\sigma(d^2\mathcal{L}_\lambda^\circ(\theta)) \cap \mathbb{R}^-$  (resp.  $\sigma(d^2\mathcal{L}_\lambda^\circ(\theta)) \cap \mathbb{R}^+$ ). Clearly,  $F_\lambda^+$  the orthogonal complement of  $F_\lambda^-$  in  $H^0$ , and the spaces  $F_\lambda^-$  and  $F_{\lambda'}^-$  are  $G$ -isomorphic if  $(\lambda - \lambda^*)(\lambda' - \lambda^*) > 0$ . Hence if  $\ell$  is the above  $(\mathcal{G}, h^*)$ -length, for  $\lambda < \lambda^* < \mu$  close to  $\lambda^*$ , the number

$$d := \ell(SH^0) - \min\{\ell(SF_\lambda^-) + \ell(SF_\mu^+), \ell(SF_\lambda^+) + \ell(SF_\mu^-)\} \quad (3.45)$$

is well-defined, and if  $\ell(SV) = c \cdot \dim V$  for every  $G$ -module  $V$  with  $V^G = \{\theta\}$  we have

$$d = |\ell(SF_\lambda^-) - \ell(SF_\mu^-)| = c|\dim F_\lambda^- - \dim F_\mu^-| = c|\dim E_\lambda - \dim E_\mu| \quad (3.46)$$

([2]). Here the final equality comes from the fact that  $P^0$  defines a  $G$ -isomorphism from  $E_\lambda$  onto  $F_\lambda^-$  (cf. [2, page 353]). Moreover, it is easy to see that  $F_\lambda^- = H_-^0$  and  $F_\lambda^+ = H_+^0$  for  $\lambda < \lambda^*$ , and  $F_\lambda^- = H_+^0$  and  $F_\lambda^+ = H_-^0$  for  $\lambda > \lambda^*$ . (3.45) and (3.46), respectively, become

$$d = \ell(SH^0) - 2 \min\{\ell(SH_-^0), \ell(SH_+^0)\}, \quad (3.47)$$

$$d = |\ell(SH_-^0) - \ell(SH_+^0)| = c|\dim H_-^0 - \dim H_+^0|. \quad (3.48)$$

Having these we may state the following partial generalization of [2, Theorem 3.1].

**Theorem 3.14.** *Under Hypothesis 3.13, if the number  $d$  in (3.45) or (3.47) is positive, then  $(\lambda^*, \theta) \in \mathbb{R} \times U$  is a bifurcation point for the equation (3.14) and one of the following alternatives occurs:*

- (i)  $(\lambda^*, \theta)$  is not an isolated solution of (3.14) in  $\{\lambda^*\} \times U$ .
- (ii) there exist left and right neighborhoods  $\Lambda_l$  and  $\Lambda_r$  of  $\lambda^*$  in  $\mathbb{R}$  and integers  $i_l, i_r \geq 0$  such that  $i_l + i_r \geq d$  and for any  $\lambda \in \Lambda_l \setminus \{\lambda^*\}$  (resp.  $\lambda \in \Lambda_r \setminus \{\lambda^*\}$ ), (3.14) has at least  $i_l$  (resp.  $i_r$ ) distinct nontrivial solution orbits, which converge to  $\theta$  in  $H$  as  $\lambda \rightarrow \lambda^*$ .

*Proof.* Suppose that (i) does not hold. Then  $\theta \in H^0$  is an isolated critical point of  $\mathcal{L}_{\lambda^*}^\circ$ . We assume that  $\theta \in H^0$  is a unique critical point in the set  $\overline{\mathcal{D}}$  of Lemma 3.10. Moreover the flow  $\varphi_\lambda$  of (3.43) may be replaced by the negative gradient one  $\chi_\lambda$  of  $\mathcal{L}_\lambda^\circ$  since we have assumed  $\mathcal{L}_\lambda^\circ$  to be of class  $C^2$ . Then we get a corresponding Lemma 3.10. For  $* = +, -$ , let  $S^* = \{z \in B_H(\theta, \epsilon) \cap H^0 \mid \chi_\lambda(s, z) \in B_H(\theta, \epsilon) \cap H^0, \forall s > 0\}$  and  $T^* = S^* \cap \partial \overline{\mathcal{D}}$ . We have the following corresponding result with the part b) of [2, Lemma 4.1].

**Lemma 3.15.** *Both  $T^+$  and  $T^-$  are  $G$ -invariant compact subset of  $\partial \overline{\mathcal{D}}$ , and also satisfy*

- 1°  $\min\{\mathcal{L}_\lambda^\circ(z) \mid z \in T^+\} > 0$  and  $\max\{\mathcal{L}_\lambda^\circ(z) \mid z \in T^-\} < 0$ ;
- 2°  $T^+$  and  $T^-$  can be deformed inside  $\overline{\mathcal{D}} \setminus \{\theta\}$  into arbitrarily small neighborhoods of  $\theta \in \overline{\mathcal{D}}$  such that  $\mathcal{L}_{\lambda^*}^\circ$  does not change sign during the deformation.
- 3°  $\ell(T^+) + \ell(T^-) \geq \ell(SH^0)$ .

Recall  $\Lambda = [\lambda^* - \delta, \lambda^* + \delta]$ . Shrinking  $\delta > 0$  (if necessary) we may assume that  $\mathcal{L}_\lambda^\circ(T^+) \subset \mathbb{R}^+$  and  $\mathcal{L}_\lambda^\circ(T^-) \subset \mathbb{R}^-$  for all  $\lambda \in \Lambda$ . Note that for each  $\lambda \neq \lambda^*$  near  $\lambda^*$ ,  $F_\lambda^+$  is the tangent space of the stable manifold of the negative gradient one  $\chi_\lambda$  because  $\theta \in H^0$  is a nondegenerate critical point of the  $C^2$  function  $\mathcal{L}_\lambda^\circ$ . We can complete proofs of corresponding results with [2, Lemmas 4.2, 4.3].  $\square$

Clearly, even if  $G = S^1$  or  $\mathbb{Z}_2$ , Theorem 3.9 cannot be included in Theorem 3.14 and the following two results.

Now consider generalizations of bifurcation results [1, §7.5]. Once  $(\mathcal{A}, h^*, I)$  is understood its  $(\mathcal{A}, h^*, I)$ -length is written as  $\ell$  below.

Under Hypothesis 3.12, let  $\varphi_\lambda$  be the flow of  $\mathcal{V}_\lambda$  given by (3.43). Assume that  $\Lambda$  is equipped with trivial  $G$ -action. By the theorem on continuous dependence of solutions of ordinary differential equations on initial values and parameters we obtain an equivariant product flow parametrized by  $\Lambda$  on  $\Lambda \times B_{H^0}(\theta, \epsilon)$ ,  $(\lambda, z, t) \rightarrow \varphi(\lambda, z, t) := (\lambda, \varphi_\lambda(z, t))$ . Clearly,  $\mathcal{V}_\lambda$  is gradient-like with Lyapunov-function  $\mathcal{L}_\lambda^\circ$ . Since  $\lambda^*$  is an isolated eigenvalue of (3.15),  $\theta \in H^0$  is an isolated critical point of  $\mathcal{L}_\lambda^\circ$  (and so an isolated invariant set of  $\varphi_\lambda$ ) for each  $\lambda$  near  $\lambda^*$  and  $\lambda \neq \lambda^*$ . Let  $\ell^u(\lambda, \theta)$  (resp.  $\ell^s(\lambda, \theta)$ ) be the exit-length (resp. entry-length) of  $\theta$  with respect to  $\varphi_\lambda$ , see [1, §7.3]. Moreover,  $\ell^u(\lambda, \theta)$  is independent of  $\lambda \in \Lambda \cap (\lambda^*, \infty)$  (resp.  $\Lambda \cap (-\infty, \lambda^*)$ ) close to  $\lambda^*$ , denoted by  $\ell_+^u$  (resp.  $\ell_-^u$ ). See [1, §7.2, §7.5] for these. By Theorems 7.10, 7.11 in [1] we immediately obtain the following two theorems.



**Theorem 3.16.** *Under Hypothesis 3.12, suppose that  $\ell_+^u \neq \ell_-^u$ . Then  $(\lambda^*, \theta)$  is a bifurcation point of  $\nabla \mathcal{L}_\lambda^\circ$ . Moreover, if  $\theta \in H^0$  is also an isolated critical point of  $\mathcal{L}_{\lambda^*}^\circ$ , then there exists  $\varepsilon > 0$  and integers  $d_l, d_r \geq 0$  with  $d_l + d_r \geq |\ell_+^u - \ell_-^u|$  so that for each  $\lambda \in (\lambda^* - \varepsilon, \lambda^*)$  respectively  $\lambda \in (\lambda^*, \lambda^* + \varepsilon)$  at least one of the following alternatives occurs:*

- (i) *there exist critical  $G$ -orbits  $Gu_\lambda^i$ ,  $i \in \mathbb{Z} \setminus \{0\}$ , of  $\mathcal{L}_\lambda^\circ$  with  $u_\lambda^i \neq \theta$ ,  $\mathcal{L}_\lambda^\circ(u_\lambda^{-i}) < \mathcal{L}_\lambda^\circ(\theta) < \mathcal{L}_\lambda^\circ(u_\lambda^i)$  for  $i \geq 1$ , and  $\mathcal{L}_\lambda^\circ(u_\lambda^i) \rightarrow \mathcal{L}_{\lambda^*}^\circ(\theta)$  as  $|i| \rightarrow \infty$ . In particular,  $\mathcal{L}_\lambda^\circ$  has infinitely many critical  $G$ -orbits. They converge to  $\theta \in H^0$  as  $\lambda \rightarrow \lambda^*$ .*
- (ii) *There exists an isolated invariant set  $S_\lambda \in B_{H^0}(\theta, \varepsilon) \setminus \{\theta\}$  with  $\ell(\mathcal{C}(S_\lambda)) \geq d_l$  respectively  $\ell(\mathcal{C}(S_\lambda)) \geq d_r$ . Moreover,  $S_\lambda$  converge to  $\theta \in H^0$  as  $\lambda \rightarrow \lambda^*$ , i.e., for any neighborhood  $W$  of  $\theta$  in  $H^0$  there exist  $\varepsilon = \varepsilon(W) > 0$  such that  $S_\lambda \in W$  if  $|\lambda - \lambda^*| < \varepsilon$ .*

The result is also true for  $G = \mathbb{Z}/p$  and  $\ell = \ell_0 + \ell_1 - 1$  as in [1, Remark 4.14].

**Theorem 3.17.** *Under Hypothesis 3.12, suppose that  $G = \mathbb{Z}/p$ ,  $p$  a prime, or  $G = S^1 \times \Gamma$ ,  $\Gamma$  a finite group, and let  $\ell$  be any of the lengths defined in [1, 4.4, 4.14 or 4.16]. If  $\theta \in H^0$  is also an isolated critical point of  $\mathcal{L}_{\lambda^*}^\circ$ , and  $\ell_-^u, \ell_+^u$  are the exit-lengths as above, then there exists  $\varepsilon > 0$  and integers  $d_l, d_r \geq 0$  with  $d_l + d_r \geq |\ell_+^u - \ell_-^u|$  such that the following holds: For each  $\lambda \in (\lambda^* - \varepsilon, \lambda^*)$  respectively  $\lambda \in (\lambda^*, \lambda^* + \varepsilon)$  there exists a compact invariant set  $S_\lambda \in B_{H^0}(\theta, \varepsilon) \setminus \{\theta\}$  with  $\ell(\mathcal{C}(S_\lambda)) \geq d_l$  respectively  $\ell(\mathcal{C}(S_\lambda)) \geq d_r$ .*

Every  $G$ -critical orbit of  $\mathcal{L}_\lambda^\circ$  produced by Theorems 3.16, 3.17 gives rise to a  $G$ -critical orbit of  $\mathcal{L}_\lambda$ , and different orbits yield different ones too. In particular,  $(\lambda^*, \theta) \in \mathbb{R} \times U$  is a bifurcation point for the equation (3.14). However, in order to understand  $\ell_+^u$  and  $\ell_-^u$  we assume that Hypothesis 3.13 is satisfied. Then the flow  $\varphi_\lambda$  may be replaced by the negative gradient one  $\chi_\lambda$  of  $\mathcal{L}_\lambda^\circ$ . By the arguments below Definition 7.1 in [1] we have  $\ell_+^u = \ell^u(\lambda, \theta) = \ell(SF_\lambda^-) = \ell(SH_+^0) = \ell(SE_\lambda)$  (resp.  $\ell_-^u = \ell^u(\lambda, \theta) = \ell(SF_\lambda^-) = \ell(SH_-^0) = \ell(SE_\lambda)$ ) if  $\lambda > \lambda^*$  (resp.  $\lambda < \lambda^*$ ) is close to  $\lambda^*$ . It follows that  $\ell(SE_{\lambda^*_-}) := \ell(SE_{\lambda^*-\rho})$  and  $\ell(SE_{\lambda^*_+}) := \ell(SE_{\lambda^*+\rho})$  are independent of small  $\rho > 0$  and that

$$|\ell_+^u - \ell_-^u| = |\ell(SH_+^0) - \ell(SH_-^0)| = |\ell(SE_{\lambda^*_+}) - \ell(SE_{\lambda^*_-})|,$$

which may be chosen as  $|\dim E_{\lambda^*_+} - \dim E_{\lambda^*_-}|$  (resp.  $\frac{1}{2}|\dim E_{\lambda^*_+} - \dim E_{\lambda^*_-}|$ ) if  $G = \mathbb{Z}/p$  with a prime  $p$  (resp.  $G = S^1 \times \Gamma$  with a finite group  $\Gamma$ ). By these, under Hypothesis 3.13, Theorem 3.16, 3.17 may be directly transformed two results about bifurcation information of (3.14) from  $(\lambda^*, \theta)$ . We here omit them.

**Remark 3.18.** (i) If  $n = \dim \Omega = 1$ , and  $\mathcal{F}, \mathcal{G}$  are defined by (1.3) under Hypothesis  $\mathfrak{F}_{2,N}$ , then they can satisfy Hypothesis 3.13 on  $W_0^{m,2}(\Omega, \mathbb{R}^N)$ ; see [39, 48]. Hence the last three theorems may be applied in this case.

(ii) The proof key of [1, Theorem 7.12] is to use the (local) center manifold theorem instead of the Lyapunov-Schmidt reduction. This method was firstly used by Chow and Lauterbach [18] in

the non-equivariant case. Our potential operators are neither strictly Fréchet differentiable at  $\theta$  nor of class  $C^1$ . Hence the center manifold theorem seems unable to be used in our situation. But, from the constructions of center manifolds by Vanderbauwhede and Iooss [64] this method is also possible if the restrictions of our potential operators to a Banach space  $X$  continuously and densely embedding in  $H$  are of class  $C^1$  as in the framework of [39, 40]; see [46].

The bifurcations in the previous theorems are all from a trivial critical orbit. Finally, let us give a result about bifurcations starting a nontrivial critical orbit.

**Hypothesis 3.19.** Under Hypothesis 2.21, let for some  $x_0 \in \mathcal{O}$  the pair  $(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}}, N\mathcal{O}(\varepsilon)_{x_0})$  satisfy the corresponding conditions with Hypothesis 1.1 with  $X = H$ . Let  $\mathcal{G} \in C^1(\mathcal{H}, \mathbb{R})$  be  $G$ -invariant, have a critical orbit  $\mathcal{O}$ , and also satisfy:

- (i) the gradient  $\nabla \mathcal{G}$  is Gâteaux differentiable near  $\mathcal{O}$ , and every derivative  $\mathcal{G}''(u)$  is also a compact linear operator;
- (ii)  $\mathcal{G}''$  are continuous at each point  $u \in \mathcal{O}$ .

The assumptions on  $\mathcal{G}$  assure that the functional  $\mathcal{L} - \lambda \mathcal{G}$ ,  $\lambda \in \mathbb{R}$ , also satisfy the conditions of Theorems 2.22, 2.23.

Let  $\lambda^*$  be an eigenvalue of

$$\mathcal{L}''(x_0)v - \lambda \mathcal{G}''(x_0)v = 0, \quad v \in T_{x_0}\mathcal{H}, \quad (3.49)$$

and  $\dim \text{Ker}(\mathcal{L}''(x_0) - \lambda^* \mathcal{G}''(x_0)) > \dim \mathcal{O}$ .

We say  $\mathcal{O}$  to be a *bifurcation  $G$ -orbit with parameter  $\lambda^*$*  of the equation

$$\mathcal{L}'(u) = \lambda \mathcal{G}'(u), \quad u \in \mathcal{H} \quad (3.50)$$

if for any  $\varepsilon > 0$  and neighborhood  $\mathcal{U}$  of  $\mathcal{O}$  in  $\mathcal{H}$  there exists a solution  $G$ -orbit  $\mathcal{O}' \neq \mathcal{O}$  in  $\mathcal{U}$  of (3.50) with some  $\lambda \in (-\varepsilon, \varepsilon)$ .

Note that the orthogonal complementary of  $T_{x_0}\mathcal{O}$  in  $T_{x_0}\mathcal{H}$ ,  $N\mathcal{O}_{x_0}$ , is an invariant subspace of  $\mathcal{L}''(x_0)$  and  $\mathcal{G}''(x_0)$ . Let  $\mathcal{L}''(x_0)^\perp$  (resp.  $\mathcal{G}''(x_0)^\perp$ ) denote the restriction self-adjoint operator of  $\mathcal{L}''(x_0)$  (resp.  $\mathcal{G}''(x_0)$ ) from  $N\mathcal{O}_{x_0}$  to itself. Then  $\mathcal{L}''(x_0)^\perp = d^2(\mathcal{L} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}})(\theta)$  and  $\mathcal{G}''(x_0)^\perp = d^2(\mathcal{G} \circ \exp|_{N\mathcal{O}(\varepsilon)_{x_0}})(\theta)$ . Suppose that  $\mathcal{L}''(x_0)^\perp$  is invertible, or equivalently  $\text{Ker}(\mathcal{L}''(x_0)) = T_{x_0}\mathcal{O}$ . Then 0 is not an eigenvalue of

$$\mathcal{L}''(x_0)^\perp v - \lambda \mathcal{G}''(x_0)^\perp v = 0, \quad v \in N\mathcal{O}_{x_0}, \quad (3.51)$$

and  $\lambda \in \mathbb{R} \setminus \{0\}$  is an eigenvalue of (3.51) if and only if  $1/\lambda$  is an eigenvalue of compact linear self-adjoint operator  $L_{x_0} := [\mathcal{L}''(x_0)^\perp]^{-1} \mathcal{G}''(x_0)^\perp \in \mathcal{L}_s(N\mathcal{O}_{x_0})$ . Hence  $\sigma(L_{x_0}) \setminus \{0\} = \{1/\lambda_n\}_{n=1}^\infty \subset \mathbb{R}$  with  $\lambda_n \rightarrow 0$ , and each  $1/\lambda_n$  has finite multiplicity. Let  $N\mathcal{O}_{x_0}^n$  be the eigensubspace corresponding to  $1/\lambda_n$  for  $n \in \mathbb{N}$ . Then  $N\mathcal{O}_{x_0}^0 = \text{Ker}(L_{x_0}) = \text{Ker}(\mathcal{G}''(x_0)^\perp)$  and

$$N\mathcal{O}_{x_0}^n = \text{Ker}(I/\lambda_n - L_{x_0}) = \text{Ker}(\mathcal{L}''(x_0)^\perp - \lambda_n \mathcal{G}''(x_0)^\perp), \quad n = 1, 2, \dots, \quad (3.52)$$

and  $N\mathcal{O}_{x_0} = \bigoplus_{n=0}^\infty N\mathcal{O}_{x_0}^n$ .

**Theorem 3.20.** *Under Hypothesis 3.19, suppose that  $\text{Ker}(\mathcal{L}''(x_0)) = T_{x_0}\mathcal{O}$  (so the operator  $\mathcal{L}''(x_0)^\perp$  is invertible) and  $\lambda^* = \lambda_{n_0}$  for some  $n_0 \in \mathbb{N}$ . Then  $\mathcal{O}$  is a bifurcation  $G$ -orbit with parameter  $\lambda^*$  of (3.50) if one of the following two conditions holds:*

a)  $\mathcal{L}''(x_0)^\perp$  is either positive definite or negative one, and

$$C_l(\mathcal{O}; \mathbb{Z}_2) \neq C_{l-\nu_{\lambda^*}}(\mathcal{O}; \mathbb{Z}_2) \quad (3.53)$$

for some  $l \in \mathbb{Z}$ , where  $\nu_{\lambda^*} = \dim N\mathcal{O}_{x_0}^{n_0}$  (is more than zero because  $\mathcal{O}$  is a degenerate critical orbit of  $\mathcal{L}_{\lambda^*}$  by the assumption in Hypothesis 3.19);

b) each  $N\mathcal{O}_{x_0}^n$  in (3.52) is an invariant subspace of  $\mathcal{L}''(x_0)^\perp$  (e.g. these are true if  $\mathcal{L}''(x_0)^\perp$  commutes with  $\mathcal{G}''(x_0)^\perp$ ), and

$$C_{l-\nu_{\lambda^*}^-}(\mathcal{O}; \mathbb{Z}_2) \neq C_{l-\nu_{\lambda^*}^+}(\mathcal{O}; \mathbb{Z}_2) \quad (3.54)$$

for some  $l \in \mathbb{Z}$ , where  $\nu_{\lambda^*}^+$  (resp.  $\nu_{\lambda^*}^-$ ) is the dimension of the positive (resp. negative) definite space of  $\mathcal{L}''(x_0)^\perp$  on  $N\mathcal{O}_{x_0}^{n_0}$ .

From the following proof it is easily seen that (3.54) may be replaced by (3.53) if we add a condition “ $\mathcal{L}''(x_0)^\perp$  is either positive definite or negative one on  $N\mathcal{O}_{x_0}^{n_0}$ ” in b).

*Proof of Theorem 3.20.* Let  $\mu_\lambda$  denote the Morse index of  $\mathcal{L}_\lambda := \mathcal{L} - \lambda\mathcal{G}$  at  $\mathcal{O}$ ,  $\lambda^* = \lambda_{n_0}$  for some  $n_0 \in \mathbb{N}$ , and let  $\nu_{\lambda^*}$  be the nullity of  $\mathcal{L}_{\lambda^*}$  at  $\mathcal{O}$ , i.e.,  $\nu_{\lambda^*} = \dim N\mathcal{O}_{x_0}^{n_0}$ . As in the proof of Theorem 3.5 we have  $\varepsilon > 0$  such that

$$\mu_\lambda = \sum_{\lambda_n < \lambda} \dim N\mathcal{O}_{x_0}^n = \begin{cases} \mu_{\lambda^*}, & \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*], \\ \mu_{\lambda^*} + \nu_{\lambda^*}, & \forall \lambda \in (\lambda^*, \lambda^* + 2\varepsilon) \end{cases} \quad (3.55)$$

if  $\mathcal{L}''(x_0)^\perp$  is positive definite,

$$\mu_\lambda = \sum_{\lambda_n > \lambda} \dim N\mathcal{O}_{x_0}^n = \begin{cases} \mu_{\lambda^*} + \nu_{\lambda^*}, & \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*), \\ \mu_{\lambda^*}, & \forall \lambda \in [\lambda^*, \lambda^* + 2\varepsilon) \end{cases} \quad (3.56)$$

if  $\mathcal{L}''(x_0)^\perp$  is negative definite, and

$$\mu_\lambda = \begin{cases} \mu_{\lambda^*} + \nu_{\lambda^*}^-, & \forall \lambda \in (\lambda^* - 2\varepsilon, \lambda^*), \\ \mu_{\lambda^*} + \nu_{\lambda^*}^+, & \forall \lambda \in (\lambda^*, \lambda^* + 2\varepsilon) \end{cases} \quad (3.57)$$

if b) holds.

Corresponding to Claim 1 in the proof of Theorem 3.5 we may also prove

**Claim 1.** *After shrinking  $\varepsilon > 0$ , if  $\{(\kappa_n, v_n)\}_{n \geq 1} \subset [\lambda^* - \varepsilon, \lambda^* + \varepsilon] \times \overline{N\mathcal{O}(\varepsilon)}$  satisfies  $\nabla \mathcal{L}_{\kappa_n}(v_n) \rightarrow \theta$  and  $\kappa_n \rightarrow \kappa_0$ , then  $\{v_n\}_{n \geq 1}$  has a convergent subsequence in  $\overline{N\mathcal{O}(\varepsilon)}$ .*

By a contradiction, assume that  $\mathcal{O}$  is not a bifurcation  $G$ -orbit with parameter  $\lambda^*$  of (3.50). Then we have  $\delta \in (0, \varepsilon]$  such that for each  $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$ ,  $\mathcal{O}$  is an unique critical orbit of  $\mathcal{L}_\lambda$  in  $\overline{N\mathcal{O}(\delta)}$ . By [14, Theorem 5.1.21] (or as in [17]) we deduce

$$C_*(\mathcal{L}_{\lambda'}, \mathcal{O}; \mathbf{K}) = C_*(\mathcal{L}_{\lambda''}, \mathcal{O}; \mathbf{K}), \quad \forall \lambda', \lambda'' \in [\lambda^* - \delta, \lambda^* + \delta]. \quad (3.58)$$

Since  $\mathcal{O}$  is a nondegenerate critical orbit of  $\mathcal{L}_\lambda$  for each  $\lambda \in [\lambda^* - \delta, \lambda^* + \delta] \setminus \{\lambda^*\}$ , as in the proof of (2.90) we derive from (2.99) that

$$C_*(\mathcal{L}_{\lambda'}, \mathcal{O}; \mathbb{Z}_2) = C_{*- \mu_{\lambda'}}(\mathcal{O}; \mathbb{Z}_2) \quad \text{and} \quad C_*(\mathcal{L}_{\lambda''}, \mathcal{O}; \mathbb{Z}_2) = C_{*- \mu_{\lambda''}}(\mathcal{O}; \mathbb{Z}_2) \quad (3.59)$$

for any  $\lambda' \in [\lambda^* - \delta, \lambda^*)$  any  $\lambda'' \in (\lambda^*, \lambda^* + \delta]$ .

If  $\mathcal{L}''(x_0)^\perp$  is positive definite, by (3.53), (3.55) and (3.59) we deduce

$$C_{l+\mu_{\lambda^*}}(\mathcal{L}_{\lambda'}, \mathcal{O}; \mathbb{Z}_2) = C_l(\mathcal{O}; \mathbb{Z}_2) \neq C_{l-\nu_{\lambda^*}}(\mathcal{O}; \mathbb{Z}_2) = C_{l+\mu_{\lambda^*}}(\mathcal{L}_{\lambda''}, \mathcal{O}; \mathbb{Z}_2)$$

for any  $\lambda' \in [\lambda^* - \delta, \lambda^*)$  any  $\lambda'' \in (\lambda^*, \lambda^* + \delta]$ . This contradicts (3.58).

Similarly, if  $\mathcal{L}''(x_0)^\perp$  is negative definite, we deduce

$$C_{l+\mu_{\lambda^*}}(\mathcal{L}_{\lambda''}, \mathcal{O}; \mathbb{Z}_2) = C_l(\mathcal{O}; \mathbb{Z}_2) \neq C_{l-\nu_{\lambda^*}}(\mathcal{O}; \mathbb{Z}_2) = C_{l+\mu_{\lambda^*}}(\mathcal{L}_{\lambda'}, \mathcal{O}; \mathbb{Z}_2)$$

for any  $\lambda' \in [\lambda^* - \delta, \lambda^*)$  any  $\lambda'' \in (\lambda^*, \lambda^* + \delta]$ , and also arrive at a contradiction to (3.58).

If b) holds, by (3.56), (3.59) and (3.54) we have

$$C_{l+\mu_{\lambda^*}}(\mathcal{L}_{\lambda'}, \mathcal{O}; \mathbb{Z}_2) = C_{l-\nu_{\lambda^*}^-}(\mathcal{O}; \mathbb{Z}_2) \neq C_{l-\nu_{\lambda^*}^+}(\mathcal{O}; \mathbb{Z}_2) = C_{l+\mu_{\lambda^*}}(\mathcal{L}_{\lambda''}, \mathcal{O}; \mathbb{Z}_2)$$

for any  $\lambda' \in [\lambda^* - \delta, \lambda^*)$  any  $\lambda'' \in (\lambda^*, \lambda^* + \delta]$ , which contradicts (3.58).  $\square$

Using Theorem 2.29 many results above can be generalized the case of bifurcations at a non-trivial critical orbit.

## Part II

# Applications to quasi-linear elliptic systems of higher order

## 4 Fundamental analytic properties for functionals $\mathcal{F}$ and $\mathcal{F}$

### 4.1 Results and preliminaries

A bounded domain  $\Omega$  in  $\mathbb{R}^n$  is said to be a *Sobolev domain* if for each integer  $0 \leq k \leq m-1$  the Sobolev space embeddings hold for it, i.e.,

$$\begin{aligned} W^{m,p}(\Omega) &\hookrightarrow W^{k,q}(\Omega) & \text{if} & \quad \frac{1}{q} \geq \frac{1}{p} - \frac{m-k}{n} > 0, \\ W^{m,p}(\Omega) &\hookrightarrow\hookrightarrow W^{k,q}(\Omega) & \text{if} & \quad \frac{1}{q} > \frac{1}{p} - \frac{m-k}{n} > 0, \\ W^{m,p}(\Omega) &\hookrightarrow\hookrightarrow W^{k,q}(\Omega) & \text{if} & \quad q < \infty, \frac{1}{p} = \frac{m-k}{n}, \\ W^{m,p}(\Omega) &\hookrightarrow\hookrightarrow C^{k,\sigma}(\overline{\Omega}) & \text{if} & \quad \frac{n}{p} < m - (k + \sigma), \quad 0 \leq \sigma < 1, \end{aligned}$$

where  $\hookrightarrow \hookrightarrow$  denotes the compact embedding. Each bounded domain  $\Omega$  in  $\mathbb{R}^n$  with suitable smooth boundary  $\partial\Omega$  is a Sobolev domain.

The key result of this section is the following theorem. Most claims of it come from the auxiliary theorem 16 in Section 3.4 of Chapter 3 in [58] or Lemma 3.2 on the page 112 of [60]. Since only partial claims were proved with  $V = W_0^{m,p}(\Omega)$  therein, for completeness we give a detailed proof of it.

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Sobolev domain,  $p \in [2, \infty)$  and let  $V$  be a closed subspace of  $W^{m,p}(\Omega)$ . Suppose that (i)-(ii) in **Hypothesis**  $\mathfrak{f}_p$  hold. Then we have*

**A).** *On  $V$  the functional  $\mathcal{F}$  in (1.8) is bounded on any bounded subset, of class  $C^1$ , and the derivative  $\mathcal{F}'(u)$  of  $\mathcal{F}$  at  $u$  is given by*

$$\langle \mathcal{F}'(u), v \rangle = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha}(x, u(x), \dots, D^m u(x)) D^{\alpha} v dx, \quad \forall v \in V. \quad (4.1)$$

*Moreover, the map  $u \rightarrow \mathcal{F}'(u)$  also maps bounded subset into bounded ones.*

**B).** *The map  $\mathcal{F}'$  is of class  $C^1$  on  $V$  if  $p > 2$ , Gâteaux differentiable on  $V$  if  $p = 2$ , and for each  $u \in V$  the derivative  $D\mathcal{F}'(u) \in \mathcal{L}(V, V^*)$  is given by*

$$\langle D\mathcal{F}'(u)v, \varphi \rangle = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx. \quad (4.2)$$

*(In the case  $p = 2$ , equivalently, the gradient map of  $\mathcal{F}$ ,  $V \ni u \mapsto \nabla \mathcal{F}(u) \in V$ , given by*

$$(\nabla \mathcal{F}(u), v)_{m,2} = \langle \mathcal{F}'(u), v \rangle \quad \forall v \in V, \quad (4.3)$$

*has a Gâteaux derivative  $D(\nabla \mathcal{F})(u) \in \mathcal{L}_s(V)$  at every  $u \in V$ .) Moreover,  $D\mathcal{F}'$  also satisfies the following properties:*

- (i)** *For every given  $R > 0$ ,  $\{D\mathcal{F}'(u) \mid \|u\|_{m,p} \leq R\}$  is bounded in  $\mathcal{L}_s(V)$ . Consequently, when  $p = 2$ ,  $F$  is on  $V$  of class  $C^{2-0}$ .*
- (ii)** *For any  $v \in V$ ,  $u_n \rightarrow u_0$  implies  $D\mathcal{F}'(u_n)v \rightarrow D\mathcal{F}'(u_0)v$  in  $V^*$ .*
- (iii)** *If  $p = 2$  and  $f(x, \xi)$  is independent of all variables  $\xi_{\alpha}$ ,  $|\alpha| = m$ , then  $V \ni u \mapsto D\mathcal{F}'(\bar{u}) \in \mathcal{L}(V, V^*)$  is continuous, i.e.,  $\mathcal{F}$  is of class  $C^2$ , and  $D(\nabla \mathcal{F})(u) : V \rightarrow V$  is completely continuous for each  $u \in V$ .*

*In addition, if (iii) in **Hypothesis**  $\mathfrak{f}_p$  is also satisfied, we further have*

**C).**  *$\mathcal{F}'$  satisfies condition  $(S)_+$ .*

**D).** Suppose  $p = 2$ . For  $u \in V$ , let  $D(\nabla \mathcal{F})(u)$ ,  $P(u)$  and  $Q(u)$  be operators in  $\mathcal{L}(V)$  defined by

$$(D(\nabla \mathcal{F})(u)v, \varphi)_{m,2} = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx, \quad (4.4)$$

$$\begin{aligned} (P(u)v, \varphi)_{m,2} &= \sum_{|\alpha|=|\beta|=m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx \\ &+ \sum_{|\alpha| \leq m-1} \int_{\Omega} D^{\alpha} v \cdot D^{\alpha} \varphi dx, \end{aligned} \quad (4.5)$$

$$\begin{aligned} (Q(u)v, \varphi)_{m,2} &= \sum_{|\alpha|+|\beta| < 2m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx \\ &- \sum_{|\alpha| \leq m-1} \int_{\Omega} D^{\alpha} v \cdot D^{\alpha} \varphi dx, \end{aligned} \quad (4.6)$$

respectively. Then  $D(\nabla \mathcal{F}) = P + Q$ , and

(i) for any  $v \in V$ , the map  $V \ni u \mapsto P(u)v \in W^{m,2}(\Omega)$  is continuous;

(ii) for every given  $R > 0$  there exist positive constants  $C(R, n, m, \Omega)$  such that

$$(P(u)v, v)_{m,2} \geq C \|v\|_{m,2}^2 \quad \forall v \in V$$

if  $u \in W^{m,2}(\Omega)$  satisfies  $\|u\|_{m,2} \leq R$ ;

(iii)  $V \ni u \mapsto Q(u) \in \mathcal{L}(V)$  is continuous, and  $Q(u) : V \rightarrow V$  is completely continuous for each  $u$ ;

(iv) for every given  $R > 0$  there exist positive constants  $C_j(R, n, m, \Omega)$ ,  $j = 1, 2$  such that

$$(D(\nabla \mathcal{F})(u)v, v)_{m,2} \geq C_1 \|v\|_{m,2}^2 - C_2 \|v\|_{m-1,2}^2 \quad \forall v \in V$$

if  $u \in V$  satisfies  $\|u\|_{m,2} \leq R$ .

This is a special case of the following result.

**Theorem 4.2.** Let  $\Omega \subset \mathbb{R}^n$  and  $p \in [2, \infty)$  be as in Theorem 4.1,  $N \geq 1$  an integer, and  $V$  a closed subspace of  $W^{m,p}(\Omega, \mathbb{R}^N)$ . Suppose that (i)-(ii) in **Hypothesis**  $\mathfrak{F}_{p,N}$  hold. Then corresponding conclusions to A) and B) in Theorem 4.1 are also true if the letters  $u, v, \mathcal{F}$  therein are replaced by  $\vec{u}, \vec{v}, \mathfrak{F}$ , respectively, and (4.1)–(4.2) are changed into

$$\langle \mathfrak{F}'(\vec{u}), \vec{v} \rangle = \sum_{i=1}^N \sum_{|\alpha| \leq m} \int_{\Omega} F_{\alpha}^i(x, \vec{u}(x), \dots, D^m \vec{u}(x)) D^{\alpha} v^i dx, \quad \forall \vec{v} \in V, \quad (4.7)$$

$$\langle D\mathfrak{F}'(\vec{u})\vec{v}, \vec{\varphi} \rangle = \sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}(x), \dots, D^m \vec{u}(x)) D^{\beta} v^j \cdot D^{\alpha} \varphi^i dx.$$

Moreover, if (iii) in **Hypothesis**  $\mathfrak{F}_{p,N}$  is also satisfied, then corresponding conclusions to C) and D) in Theorem 4.1 still remain true if the letters  $u, v, \mathcal{F}$  therein are replaced by  $\vec{u}, \vec{v}, \mathfrak{F}$ , respectively, and (4.4)–(4.6) are changed into

$$\begin{aligned}
(D(\nabla \mathfrak{F})(\vec{u})\vec{v}, \vec{\varphi})_{m,2} &= \sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}(x), \dots, D^m \vec{u}(x)) D^{\beta} v^j \cdot D^{\alpha} \varphi^i dx, \\
(P(\vec{u})\vec{v}, \vec{\varphi})_{m,2} &= \sum_{i,j=1}^N \sum_{|\alpha|=|\beta|=m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}(x), \dots, D^m \vec{u}(x)) D^{\beta} v^j \cdot D^{\alpha} \varphi^i dx \\
&\quad + \sum_{i=1}^N \sum_{|\alpha| \leq m-1} \int_{\Omega} D^{\alpha} v^i \cdot D^{\alpha} \varphi^i dx, \\
(Q(\vec{u})\vec{v}, \vec{\varphi})_{m,2} &= \sum_{i,j=1}^N \sum_{|\alpha|+|\beta| < 2m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}(x), \dots, D^m \vec{u}(x)) D^{\beta} v^j \cdot D^{\alpha} \varphi^i dx \\
&\quad - \sum_{i=1}^N \sum_{|\alpha| \leq m-1} \int_{\Omega} D^{\alpha} v^i \cdot D^{\alpha} \varphi^i dx.
\end{aligned}$$

Theorem 4.2 can be proved as that of Theorem 4.1, only more terms are added or estimated in each step. For the sake of simplicity, we only prove Theorem 4.1. To this goal the following preliminary results are needed.

**Proposition 4.3.** *For the function  $\mathfrak{g}_1$  in **Hypothesis**, let continuous positive nondecreasing functions  $\mathfrak{g}_k : [0, \infty) \rightarrow \mathbb{R}$ ,  $k = 3, 4, 5$ , be given by*

$$\begin{aligned}
\mathfrak{g}_3(t) &:= 1 + \mathfrak{g}_1(t)[t^2 M(m) + t(M(m) + 1)^2] + \mathfrak{g}_1(t)t(M(m) + 1) + \mathfrak{g}_1(t)(M(m) + 1)^2, \\
\mathfrak{g}_4(t) &:= \mathfrak{g}_1(t)t + \mathfrak{g}_1(t) \quad \text{and} \quad \mathfrak{g}_5(t) := (M(m) + 1)\mathfrak{g}_1(t)(t + 1).
\end{aligned}$$

Then (ii) in **Hypothesis**  $\mathfrak{f}_p$  implies that for all  $(x, \xi)$ ,

$$\begin{aligned}
|f(x, \xi)| &\leq |f(x, 0)| + |\xi_{\circ}| \sum_{|\alpha| < m-n/p} |f_{\alpha}(x, 0)| + \sum_{m-n/p \leq |\alpha| \leq m} |f_{\alpha}(x, 0)|^{q_{\alpha}} \\
&\quad + \mathfrak{g}_3(|\xi_{\circ}|) \left( 1 + \sum_{m-n/p \leq |\alpha| \leq m} |\xi_{\alpha}|^{p_{\alpha}} \right), \tag{4.8}
\end{aligned}$$

$$\begin{aligned}
|f_{\alpha}(x, \xi)| &\leq |f_{\alpha}(x, 0)| + \mathfrak{g}_4(|\xi_{\circ}|) \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_{\gamma}|^{p_{\gamma}} \right)^{p_{\alpha\beta}} \\
&\quad + \mathfrak{g}_4(|\xi_{\circ}|) \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_{\gamma}|^{p_{\gamma}} \right)^{p_{\alpha\beta}} |\xi_{\beta}|; \tag{4.9}
\end{aligned}$$

for the latter we further have

$$|f_{\alpha}(x, \xi)| \leq |f_{\alpha}(x, 0)| + \mathfrak{g}_5(|\xi_{\circ}|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_{\gamma}|^{p_{\gamma}} \right), \tag{4.10}$$

if  $|\alpha| < m - n/p$ , and

$$|f_\alpha(x, \xi)| \leq |f_\alpha(x, 0)| + \mathfrak{g}_5(|\xi_\circ|) + \mathfrak{g}_5(|\xi_\circ|) \left( \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha} \quad (4.11)$$

if  $m - n/p \leq |\alpha| \leq m$ .

Its proof will be given in Appendix A. The following standard result concerning the continuity of the Nemytski operator (cf. [7, Lemma 3.2] and [60, Proposition 1.1, page 3]) will be used many times.

**Proposition 4.4.** *Let  $G$  be a measurable set of positive measure in  $\mathbb{R}^N$  and let  $f : G \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfy the following conditions:*

- (a)  $f(x, \xi_1, \dots, \xi_N)$  is continuous in  $(\xi_1, \dots, \xi_N)$  for almost all  $x \in G$ ;
- (b)  $f(x, \xi_1, \dots, \xi_N)$  is measurable in  $x$  for any fixed  $(\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ ;
- (c) there exist positive numbers  $C, 1 < p, p_1, \dots, p_N < \infty$  and a function  $g \in L^p(G)$  such that

$$|f(x, \xi_1, \dots, \xi_N)| \leq C \sum_{i=1}^N |\xi_i|^{\frac{p_i}{p}} + g(x), \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N.$$

Then the Nemytskii operator  $F : \prod_{i=1}^N L_{p_i}(G) \rightarrow L_p(G)$  defined by the formula

$$F(u_1, \dots, u_N)(x) = f(x, u_1(x), \dots, u_N(x))$$

is bounded (i.e. mapping bounded sets into bounded sets) and continuous.

The following basic inequalities are standard.

**Lemma 4.5.** (i) *There exists a positive constant  $C$  only depending on  $p \geq 2$  such that*

$$\int_0^1 (1 + |ta + (1-t)b|)^{p-2} dt \geq C(1 + |a| + |b|)^{p-2}, \quad \forall a, b \in \mathbb{R}.$$

(ii)  $(|x_1| + \dots + |x_n|)^q \leq |x_1|^q + \dots + |x_n|^q$  for any  $q \in (0, 1)$  and numbers  $x_j, j = 1, \dots$ .

We shall prove Theorem 4.1 in four subsections. Clearly, it suffices to prove the case  $V = W^{m,p}(\Omega)$ .

## 4.2 Proof for A) of Theorem 4.1

**Step 1.** *Prove the continuity of  $\mathcal{F}$ .*

By (4.8) it is easy to see that the functional  $\mathcal{F}$  in (1.8) is well-defined on  $W^{m,p}(\Omega)$  and is bounded on any bounded subset of  $W^{m,p}(\Omega)$ .



Next, we prove that  $\mathcal{F}$  is continuous at a fixed  $u_0 \in W^{m,p}(\Omega)$ . For  $\delta > 0$  let  $B(u_0, \delta) = \{u \in W^{m,p}(\Omega) \mid \|u - u_0\|_{m,p} \leq \delta\}$ . By the Sobolev embedding theorem there exist  $R = R(u_0) > 0$  such that  $\sup\{|D^\alpha u(x)| : |\alpha| < m - n/p, x \in \Omega\} \leq R$  for all  $u \in B(u_0, 1)$ . Take a continuous function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\chi(t) = t \ \forall |t| \leq 2R$ ,  $\chi(t) = \pm 3R \ \forall \pm t \geq 3R$ , and  $|\chi(t)| \leq 3R$ . Define a function  $\tilde{f} : \overline{\Omega} \times \mathbb{R}^{M(m)} \rightarrow \mathbb{R}$ ,  $(x, \xi) \mapsto \tilde{f}(x, \xi) = f(x, \tilde{\xi})$ , where  $\tilde{\xi}_\alpha = \chi(\xi_\alpha)$  if  $|\alpha| < m - n/p$ , and  $\tilde{\xi}_\alpha = \xi_\alpha$  if  $m - n/p \leq |\alpha| \leq m$ . Then  $\tilde{0} = 0$ ,  $|\tilde{\xi}_\alpha| \leq 3M(m)R$ . It follows that  $\tilde{f}$  also satisfies the Caratheodory condition, and therefore (4.8) leads to

$$\begin{aligned} |\tilde{f}(x, \xi)| &\leq |f(x, 0)| + 3M(m)R \sum_{|\alpha| < m-n/p} |f_\alpha(x, 0)| + \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, 0)|^{q_\alpha} \\ &\quad + \mathfrak{g}_3(3M(m)R) \left( 1 + \sum_{m-n/p \leq |\alpha| \leq m} |\xi_\alpha|^{p_\alpha} \right). \end{aligned} \quad (4.12)$$

Let the functional  $\tilde{\mathcal{F}} : W^{m,p}(\Omega) \rightarrow \mathbb{R}$  be defined by

$$\tilde{\mathcal{F}}(u) = \int_{\Omega} \tilde{f}(x, u(x), \dots, D^m u(x)) dx.$$

Clearly, it is equal to  $\mathcal{F}$  on ball  $B(u_0, 1)$ . Hence we only need to prove that  $\tilde{\mathcal{F}}$  is continuous at  $u_0$ . This can follow from (4.12) and Proposition 4.4.

**Step 2.** *Prove the  $C^1$ -smoothness of  $\mathcal{F}$ .*

Fix  $u, \varphi \in W^{m,p}(\Omega)$ . Let  $C_{u,\varphi} = \sup_{k < m - \frac{n}{p}} [\|u\|_{C^k} + \|\varphi\|_{C^k}]$ . For any  $t \in [-1, 1] \setminus \{0\}$  and a.a.  $x \in \Omega$ , using the intermediate value theorem and (4.10)-(4.11) we deduce

$$\begin{aligned} &\left| \frac{1}{t} [f(x, u(x) + t\varphi(x), \dots, D^m u(x) + tD^m \varphi(x)) - f(x, u(x), \dots, D^m u(x))] \right| \\ &\leq \sup_{0 \leq \vartheta \leq 1} \sum_{|\alpha| < m-n/p} |f_\alpha(x, u(x) + \vartheta t\varphi(x), \dots, D^m u(x) + \vartheta tD^m \varphi(x))| \cdot |D^\alpha \varphi(x)| \\ &\quad + \sup_{0 \leq \vartheta \leq 1} \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, u(x) + \vartheta t\varphi(x), \dots, D^m u(x) + \vartheta tD^m \varphi(x))| \cdot |D^\alpha \varphi(x)| \\ &\leq C_{u,\varphi} \sup_{0 \leq \vartheta \leq 1} \sum_{|\alpha| < m-n/p} \left[ |f_\alpha(x, 0)| + \mathfrak{g}_5(C_{u,\varphi}) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x) + \vartheta tD^\gamma \varphi(x)|^{p_\gamma} \right) \right] \\ &\quad + \sup_{0 \leq \vartheta \leq 1} \sum_{m-n/p \leq |\alpha| \leq m} \left[ |f_\alpha(x, 0)| + \mathfrak{g}_5(C_{u,\varphi}) \right. \\ &\quad \quad \left. + \mathfrak{g}_5(C_{u,\varphi}) \left( \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x) + \vartheta tD^\gamma \varphi(x)|^{p_\gamma} \right)^{1/q_\alpha} \right] \cdot |D^\alpha \varphi(x)| \\ &\leq C_{u,\varphi} \sum_{|\alpha| < m-n/p} \left[ |f_\alpha(x, 0)| + \mathfrak{g}_5(C_{u,\varphi}) \right. \\ &\quad \quad \left. + \mathfrak{g}_5(C_{u,\varphi}) \left( \sum_{m-n/p \leq |\gamma| \leq m} 2^{p_\gamma} [|D^\gamma u(x)|^{p_\gamma} + |D^\gamma \varphi(x)|^{p_\gamma}] \right) \right] \\ &\quad + \sum_{m-n/p \leq |\alpha| \leq m} [|f_\alpha(x, 0)| + \mathfrak{g}_5(C_{u,\varphi})] \cdot |D^\alpha \varphi(x)| \\ &\quad + \mathfrak{g}_5(C_{u,\varphi}) \sum_{m-n/p \leq |\alpha| \leq m} \left( \sum_{m-n/p \leq |\gamma| \leq m} 2^{p_\gamma} [|D^\gamma u(x)|^{p_\gamma} + |D^\gamma \varphi(x)|^{p_\gamma}] \right)^{1/q_\alpha} \cdot |D^\alpha \varphi(x)|. \end{aligned}$$

It follows from the assumptions on  $p_{\alpha\beta}$  that the right side is integrable and thus from the Lebesgue dominated convergence theorem that the functional  $\mathcal{F}$  is Gâteaux differentiable. Moreover, the Gâteaux differential of  $\mathcal{F}$  at  $u$ ,  $D\mathcal{F}(u) \in [W^{m,p}(\Omega)]^*$ , is given by

$$D\mathcal{F}(u)\varphi = \langle D\mathcal{F}(u), \varphi \rangle = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha}(x, u(x), \dots, D^m u(x)) D^{\alpha} \varphi(x) dx.$$

Let  $D_{\alpha}\mathcal{F}(u) \in [W^{m,p}(\Omega)]^*$  be defined by

$$\langle D_{\alpha}\mathcal{F}(u), \varphi \rangle = \int_{\Omega} f_{\alpha}(x, u(x), \dots, D^m u(x)) D^{\alpha} \varphi(x) dx. \quad (4.13)$$

*Claim 1.* The map  $D_{\alpha}\mathcal{F} : W^{m,p}(\Omega) \rightarrow [W^{m,p}(\Omega)]^*$  is continuous.

- Case  $|\alpha| < m - n/p$ .

Then  $\|D^{\alpha}\varphi\|_{C^0} \leq C\|\varphi\|_{m,p}$ ,  $\forall \varphi \in W^{m,p}(\Omega)$ , where  $C > 0$  is a constant coming from the Sobolev embedding theorem. Fix  $u \in W^{m,p}(\Omega)$ . For any  $v \in B(u, 1)$ , we have

$$\begin{aligned} \|D_{\alpha}\mathcal{F}(v) - D_{\alpha}\mathcal{F}(u)\| &= \sup_{\|\varphi\|_{m,p}=1} |\langle D_{\alpha}\mathcal{F}(v) - D_{\alpha}\mathcal{F}(u), \varphi \rangle| \\ &\leq \sup_{\|\varphi\|_{m,p}=1} \int_{\Omega} |f_{\alpha}(x, v(x), \dots, D^m v(x)) - f_{\alpha}(x, u(x), \dots, D^m u(x))| \cdot |D^{\alpha} \varphi(x)| dx \\ &\leq \sup_{\|\varphi\|_{m,p}=1} \|D^{\alpha} \varphi\|_{C^0} \int_{\Omega} |f_{\alpha}(x, v(x), \dots, D^m v(x)) - f_{\alpha}(x, u(x), \dots, D^m u(x))| dx \\ &\leq C \int_{\Omega} |f_{\alpha}(x, v(x), \dots, D^m v(x)) - f_{\alpha}(x, u(x), \dots, D^m u(x))| dx. \end{aligned} \quad (4.14)$$

Because of (4.10), by a standard method as in Step 1 we may also assume that for some constant  $C' > 0$ ,

$$|f_{\alpha}(x, \xi)| \leq |f_{\alpha}(x, 0)| + C' \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_{\gamma}|^{p_{\gamma}} \right), \quad \forall (x, \xi).$$

Note that  $f_{\alpha}(\cdot, 0) \in L^1(\Omega)$ . Using Proposition 4.4 we deduce that

$$\prod_{|\beta| < n/p} L^1(\Omega) \times \prod_{m - \frac{n}{p} \leq |\beta| \leq m} L^{p_{\beta}}(\Omega) \rightarrow L^1(\Omega), \quad \mathbf{u} = \{u_{\beta} : |\beta| \leq m\} \rightarrow f_{\alpha}(\cdot, \mathbf{u})$$

is continuous, which implies the continuity of the map  $W^{m,p}(\Omega) \ni u \mapsto f_{\alpha}(\cdot, u, \dots, D^m u) \in L^1(\Omega)$ . The latter claim and (4.14) yields that  $\|D_{\alpha}\mathcal{F}(v) - D_{\alpha}\mathcal{F}(u)\| \rightarrow 0$  as  $\|v - u\|_{m,p} \rightarrow 0$ .

That is,  $D_{\alpha}\mathcal{F}$  is continuous in this case.

- Case  $m - n/p \leq |\alpha| \leq m$ . Then we have

$$\begin{aligned} \|D_{\alpha}\mathcal{F}(u) - D_{\alpha}\mathcal{F}(v)\| &= \sup_{\|\varphi\|_{m,p}=1} |\langle D_{\alpha}\mathcal{F}(u) - D_{\alpha}\mathcal{F}(v), \varphi \rangle| \\ &\leq \sup_{\|\varphi\|_{m,p}=1} \int_{\Omega} |f_{\alpha}(x, u(x), \dots, D^m u(x)) - f_{\alpha}(x, v(x), \dots, D^m v(x))| \cdot |D^{\alpha} \varphi(x)| dx \\ &\leq \sup_{\|\varphi\|_{m,p}=1} \|D^{\alpha} \varphi\|_{p_{\alpha}} \left( \int_{\Omega} |f_{\alpha}(x, u(x), \dots, D^m u(x)) - f_{\alpha}(x, v(x), \dots, D^m v(x))|^{q_{\alpha}} dx \right)^{1/q_{\alpha}} \\ &\leq C \left( \int_{\Omega} |f_{\alpha}(x, u(x), \dots, D^m u(x)) - f_{\alpha}(x, v(x), \dots, D^m v(x))|^{q_{\alpha}} dx \right)^{1/q_{\alpha}}. \end{aligned}$$

Since  $v \in B(u, 1)$ , as in Step 1, by (4.11) we may also assume that for some constant  $C > 0$ ,

$$|f_\alpha(x, \xi)| \leq |f_\alpha(x, 0)| + C \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma/q_\alpha} \right), \quad \forall (x, \xi).$$

Note that  $f_\alpha(\cdot, 0) \in L^{q_\alpha}(\Omega)$  with  $q_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ . We derive from Proposition 4.4 that

$$\prod_{|\beta| < n/p} L^1(\Omega) \times \prod_{m - \frac{n}{p} \leq |\beta| \leq m} L^{p_\beta}(\Omega) \rightarrow L^{q_\alpha}(\Omega), \quad \mathbf{u} = \{u_\beta : |\beta| \leq m\} \rightarrow f_\alpha(\cdot, \mathbf{u})$$

is continuous. This leads to the continuity of the maps  $W^{m,p}(\Omega) \ni u \mapsto f_\alpha(\cdot, u, \dots, D^m u) \in L^{q_\alpha}(\Omega)$  and hence  $D_\alpha \mathcal{F}$  as above.

To sum up, the map  $D\mathcal{F} : W^{m,p}(\Omega) \rightarrow [W^{m,p}(\Omega)]^*$  is continuous. As usual this implies that  $\mathcal{F}$  has the Fréchet derivative  $\mathcal{F}'(u) = D\mathcal{F}(u)$  at each point  $u \in W^{m,p}(\Omega)$  and thus is of class  $C^1$ .

Moreover, from the above proof we see that for any  $u \in W^{m,p}(\Omega)$ ,

$$\|D_\alpha \mathcal{F}(u)\| = \sup_{\|\varphi\|_{m,p}=1} |\langle D_\alpha \mathcal{F}(u), \varphi \rangle| \leq C \int_\Omega |f_\alpha(x, u(x), \dots, D^m u(x))| dx$$

if  $|\alpha| < m - n/p$ , and

$$\|D_\alpha \mathcal{F}(u)\| = \sup_{\|\varphi\|_{m,p}=1} |\langle D_\alpha \mathcal{F}(u), \varphi \rangle| \leq C \left( \int_\Omega |f_\alpha(x, u(x), \dots, D^m u(x))|^{q_\alpha} dx \right)^{1/q_\alpha}$$

if  $m - n/p \leq |\alpha| \leq m$ . It follows from these and (4.10)-(4.11) that  $\mathcal{F}'$  maps a bounded subset into a bounded set.

### 4.3 Proof for B) of Theorem 4.1

**Step 1.** Prove that the right side of (4.2) determines an operator in  $\mathcal{L}(W^{m,p}(\Omega), [W^{m,p}(\Omega)]^*)$ .

By (1.5) we deduce that the right side of (4.2) satisfies

$$\begin{aligned} & \left| \sum_{|\alpha|, |\beta| \leq m} \int_\Omega f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha \varphi dx \right| \\ & \leq \sup_{k < m - n/p} \mathfrak{g}_1(\|u\|_{C^k}) \sum_{|\alpha|, |\beta| \leq m} \int_\Omega \left( 1 + \sum_{m - n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma} \right)^{p_{\alpha\beta}} |D^\beta v(x)| \cdot |D^\alpha \varphi(x)| dx. \end{aligned}$$

It suffices to prove that there exists a constant  $C = C(u, \alpha, \beta)$  such that

$$\int_\Omega \left( 1 + \sum_{m - n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma} \right)^{p_{\alpha\beta}} |D^\beta v(x)| \cdot |D^\alpha \varphi(x)| dx \leq C \|v\|_{m,p} \|\varphi\|_{m,p}. \quad (4.15)$$

Case  $|\alpha| = |\beta| = m$ . Then  $p_{\alpha\beta} = 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta}$  and thus the Hölder inequality leads to

$$\begin{aligned} & \int_\Omega \left( 1 + \sum_{m - n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma} \right)^{p_{\alpha\beta}} |D^\beta v(x)| \cdot |D^\alpha \varphi(x)| dx \\ & \leq \left( \int_\Omega \left( 1 + \sum_{m - n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma} \right)^{p_{\alpha\beta}} \right)^{1/p_\beta} \left( \int_\Omega |D^\beta v(x)|^{p_\beta} \right)^{1/p_\beta} \left( \int_\Omega |D^\alpha \varphi(x)|^{p_\alpha} \right)^{1/p_\alpha} \\ & \leq C \|v\|_{m,p} \|\varphi\|_{m,p}. \end{aligned}$$

Case  $m - n/p \leq |\alpha| \leq m$ ,  $|\beta| < m - n/p$ . Then  $p_{\alpha\beta} = 1 - \frac{1}{p_\alpha}$  and  $\sup_x |D^\beta v(x)| \leq C(m, n, p) \|v\|_{m,p}$ . It follows from these that

$$\begin{aligned}
& \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right)^{p_{\alpha\beta}} |D^\beta v(x)| \cdot |D^\alpha \varphi(x)| dx \\
& \leq C(m, n, p) \|v\|_{m,p} \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right)^{p_{\alpha\beta}} \cdot |D^\alpha \varphi(x)| dx \\
& \leq C(m, n, p) \|v\|_{m,p} \left( \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right)^{p_{\alpha\beta}} \left( \int_{\Omega} |D^\alpha \varphi(x)|^{p_\alpha} \right)^{1/p_\alpha} dx \right) \\
& \leq C \|v\|_{m,p} \|\varphi\|_{m,p}.
\end{aligned}$$

Case  $m - n/p \leq |\beta| \leq m$ ,  $|\alpha| < m - n/p$ . The proof is the same as the last case.

Case  $|\alpha|, |\beta| < m - n/p$ . We have  $p_{\alpha\beta} = 1$ ,  $\sup_x |D^\beta v(x)| \leq C(m, n, p) \|v\|_{m,p}$  and  $\sup_x |D^\alpha \varphi(x)| \leq C(m, n, p) \|\varphi\|_{m,p}$ . Hence

$$\begin{aligned}
& \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right)^{p_{\alpha\beta}} |D^\beta v(x)| \cdot |D^\alpha \varphi(x)| dx \\
& \leq C(m, n, p) \|v\|_{m,p} \|\varphi\|_{m,p} \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right) dx \\
& \leq C \|v\|_{m,p} \|\varphi\|_{m,p}.
\end{aligned}$$

Case  $|\alpha|, |\beta| \geq m - n/p$ ,  $|\alpha| + |\beta| < 2m$ . Then  $0 < p_{\alpha\beta} < 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta}$ . Let  $q_{\alpha\beta} > 1$  be determined by  $q_{\alpha\beta} + \frac{1}{p_{\alpha\beta}} + \frac{1}{p_\alpha} + \frac{1}{p_\beta} = 1$ . Using the Hölder inequality we get

$$\begin{aligned}
& \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right)^{p_{\alpha\beta}} |D^\beta v(x)| \cdot |D^\alpha \varphi(x)| dx \\
& \leq |\Omega|^{1/q_{\alpha\beta}} \left( \int_{\Omega} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |D^\gamma u(x)|^{p_\gamma}\right)^{p_{\alpha\beta}} \left( \int_{\Omega} |D^\beta v(x)|^{p_\beta} \right)^{1/p_\beta} \left( \int_{\Omega} |D^\alpha \varphi(x)|^{p_\alpha} \right)^{1/p_\alpha} dx \right) \\
& \leq C \|v\|_{m,p} \|\varphi\|_{m,p}.
\end{aligned}$$

In summary, (4.15) is proved. Hence the right side of (4.2) determines an operator  $A(u) \in \mathcal{L}(W^{m,p}(\Omega), [W^{m,p}(\Omega)]^*)$  by

$$\langle A(u)v, \varphi \rangle = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha \varphi dx. \quad (4.16)$$

In particular, each term  $\sum_{|\beta| \leq m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha \varphi dx$  determines an operator  $A_\alpha(u) \in \mathcal{L}(W^{m,p}(\Omega), [W^{m,p}(\Omega)]^*)$  by

$$\langle A_\alpha(u)v, \varphi \rangle = \sum_{|\beta| \leq m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha \varphi dx. \quad (4.17)$$

**Step 2.** Prove that the map  $D_\alpha \mathcal{F} : W^{m,p}(\Omega) \rightarrow [W^{m,p}(\Omega)]^*$  defined by (4.13) is of class  $C^1$  for  $p > 2$  or  $p = 2$  and  $|\alpha| < m$ , but only Gâteaux differentiable for  $p = 2$  and  $|\alpha| = m$ .

For any  $t \in [-1, 1] \setminus \{0\}$  and  $v \in W^{m,p}(\Omega)$  we derive from (4.13) and (4.17) that

$$\begin{aligned}
& \left\langle \frac{1}{t} [D_\alpha \mathcal{F}(u + tv) - D_\alpha \mathcal{F}(u)], \varphi \right\rangle - \langle A_\alpha(u)v, \varphi \rangle \\
&= \int_\Omega \frac{1}{t} [f_\alpha(x, u(x) + tv(x), \dots, D^m u(x) + tD^m v(x)) - f_\alpha(x, u(x), \dots, D^m u(x))] D^\alpha \varphi(x) dx \\
&\quad - \sum_{|\beta| \leq m} \int_\Omega f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha \varphi dx \\
&= \sum_{|\beta| \leq m} \int_\Omega \int_0^1 f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) D^\beta v \cdot D^\alpha \varphi ds dx \\
&\quad - \sum_{|\beta| \leq m} \int_\Omega f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha \varphi dx \\
&= \sum_{|\beta| \leq m} \int_\Omega \int_0^1 [f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \\
&\quad - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))] D^\beta v \cdot D^\alpha \varphi ds dx \\
&= \sum_{|\beta| \leq m} I_{\alpha\beta}. \tag{4.18}
\end{aligned}$$

Firstly, we consider the case  $p > 2$ .

• *Case  $|\alpha| = |\beta| = m$ .* Then  $p_{\alpha\beta} = 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta} \in (0, 1)$  and we obtain

$$\begin{aligned}
|I_{\alpha\beta}| &\leq \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\
&\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \left( \int_\Omega |D^\beta v|^{p_\beta} \right)^{1/p_\beta} \left( \int_\Omega |D^\alpha \varphi|^{p_\alpha} \right)^{1/p_\alpha} \\
&\leq C \|v\|_{m,p} \|\varphi\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\
&\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}}.
\end{aligned}$$

• *Case  $m - n/p \leq |\alpha| \leq m$ ,  $|\beta| < m - n/p$ .* Then  $p_{\alpha\beta} = 1 - \frac{1}{p_\alpha}$  and  $\sup_x |D^\beta v(x)| \leq C(m, n, p) \|v\|_{m,p}$ . We can also deduce

$$\begin{aligned}
|I_{\alpha\beta}| &\leq C \|v\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\
&\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \left( \int_\Omega |D^\alpha \varphi|^{p_\alpha} \right)^{1/p_\alpha} \\
&\leq C \|v\|_{m,p} \|\varphi\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\
&\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}}.
\end{aligned}$$

• *Case*  $m - n/p \leq |\beta| \leq m$ ,  $|\alpha| < m - n/p$ . Then  $p_{\beta\alpha} = 1 - \frac{1}{p_\beta}$  and  $\sup_x |D^\alpha \varphi(x)| \leq C(m, n, p) \|\varphi\|_{m,p}$ . We can also deduce

$$\begin{aligned} |I_{\alpha\beta}| &\leq C \|\varphi\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ &\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\beta\alpha}} \right)^{p_{\beta\alpha}} \left( \int_\Omega |D^\beta v|^{p_\beta} \right)^{1/p_\alpha} \\ &\leq C \|v\|_{m,p} \|\varphi\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ &\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\beta\alpha}} \right)^{p_{\beta\alpha}}. \end{aligned}$$

• *Case*  $|\alpha|, |\beta| \geq m - n/p$ ,  $|\alpha| + |\beta| < 2m$ . Then  $0 < p_{\alpha\beta} < 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta}$ . Let  $q_{\alpha\beta} > 1$  be determined by  $q_{\alpha\beta} + \frac{1}{p_{\alpha\beta}} + \frac{1}{p_\alpha} + \frac{1}{p_\beta} = 1$ . Then

$$\begin{aligned} &\int_0^1 ds \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \\ &\quad - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))| \cdot |D^\beta v| \cdot |D^\alpha \varphi| ds dx \\ &\leq |\Omega|^{1/q_{\alpha\beta}} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ &\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \left( \int_\Omega |D^\beta v|^{p_\beta} \right)^{1/p_\beta} \left( \int_\Omega |D^\alpha \varphi|^{p_\alpha} \right)^{1/p_\alpha} \\ &\leq C \|v\|_{m,p} \|\varphi\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ &\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}}. \end{aligned}$$

• *Case*  $|\alpha| < m - n/p$ ,  $|\beta| < m - n/p$ . Then  $p_{\beta\alpha} = 1$ ,  $\sup_x |D^\beta v(x)| \leq C(m, n, p) \|\varphi\|_{m,p}$  and  $\sup_x |D^\alpha \varphi(x)| \leq C(m, n, p) \|\varphi\|_{m,p}$ . We can also deduce

$$\begin{aligned} |I_{\alpha\beta}| &\leq C \|v\|_{m,p} \|\varphi\|_{m,p} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ &\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))| \right). \end{aligned}$$

Summarizing the above five cases, by (4.18) we obtain that for any  $t \in [-1, 1] \setminus \{0\}$ ,

$$\begin{aligned} &\| [D_\alpha \mathcal{F}(u + tv) - D_\alpha \mathcal{F}(u)]/t - A_\alpha(u)v \| \\ &\leq C \|v\|_{m,p} \sum_{|\beta| \leq m} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ &\quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}}. \end{aligned} \quad (4.19)$$

Fix  $u, v \in W^{m,p}(\Omega)$ . Because of (1.5), after treating as in Step 1 of Section 4.2 we assume that for some constant  $C = C_{u,v} > 0$  and all  $(x, \xi)$ ,

$$|f_{\alpha\beta}(x, \xi)| \leq C \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} \leq C \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma p_{\alpha\beta}} \right). \quad (4.20)$$

Then we derive from Proposition 4.4 that

$$\prod_{|\beta| < n/p} L^1(\Omega) \times \prod_{m - \frac{n}{p} \leq |\gamma| \leq m} L^{p_\gamma}(\Omega) \rightarrow L^{1/p_{\alpha\beta}}(\Omega), \mathbf{u} = \{u_\gamma : |\gamma| \leq m\} \rightarrow f_{\alpha\beta}(\cdot, \mathbf{u})$$

is continuous. This implies the continuity of the map

$$W^{m,p}(\Omega) \ni w \mapsto f_{\alpha\beta}(\cdot, w, \dots, D^m w) \in L^{1/p_{\alpha\beta}}(\Omega).$$

It follows from (4.19) that

$$\lim_{t \rightarrow 0} \|[D_\alpha \mathcal{F}(u + tv) - D_\alpha \mathcal{F}(u)]/t - A_\alpha(u)v\| = 0, \quad (4.21)$$

Namely,  $D_\alpha \mathcal{F}$  has Gâteaux derivative  $A_\alpha(u)$  at  $u$ .

If  $v$  is allowed to varies in the ball  $B(u, 1) \subset W^{m,p}(\Omega)$ , then we may assume that (4.20) also holds for another constant  $C = C_{u,1} > 0$ , and thus that  $C$  in (4.19) may be changed into  $C_{u,1}$ . Taking  $t = 1$  and letting  $\|v\|_{m,p} \rightarrow 0$  in (4.19) we get that

$$\|D_\alpha \mathcal{F}(u + v) - D_\alpha \mathcal{F}(u) - A_\alpha(u)v\| = o(\|v\|_{m,p}). \quad (4.22)$$

That is,  $D_\alpha \mathcal{F}$  has Fréchet derivative  $A_\alpha(u)$  at  $u$ .

Moreover, using a similar method to the above one we can prove: for any  $u, v \in W^{m,p}(\Omega)$ , we have  $C = C(m, n, p, \Omega) > 0$  such that

$$\begin{aligned} & \|A_\alpha(u) - A_\alpha(v)\| \\ & \leq C \sum_{|\beta| \leq m} \left( \int_\Omega |f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) - f_{\alpha\beta}(x, v(x), \dots, D^m v(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}}. \end{aligned} \quad (4.23)$$

This shows that  $A_\alpha$  is continuous. Hence for  $p > 2$  we have proved:

$A$  is continuous, and  $\mathcal{F}$  is of class  $C^2$  on  $W^{m,p}(\Omega)$ .

Next, we consider the case  $p = 2$ .

If  $|\alpha| + |\beta| < 2m$  the above arguments also work, and so  $A_\alpha$  is of class  $C^1$ .

If  $|\alpha| = |\beta| = m$ , then  $p_{\alpha\beta} = 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta} = 0$  (since  $p_\alpha = p_\beta = 2$ ). We can only obtain

$$\begin{aligned} & \int_0^1 ds \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \\ & \quad - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))| \cdot |D^\beta v| \cdot |D^\alpha \varphi| ds dx \\ & \leq \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ & \quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^2 |D^\beta v|^2 \right)^{1/2} \left( \int_\Omega |D^\alpha \varphi|^2 \right)^{1/2} \\ & \leq C \|\varphi\|_{m,2} \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ & \quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^2 |D^\beta v|^2 \right)^{1/2}. \end{aligned}$$

And therefore

$$\begin{aligned} & \| [D_\alpha \mathcal{F}(u + tv) - D_\alpha \mathcal{F}(u)]/t - A_\alpha(u)v \| \\ & \leq C \int_0^1 ds \left( \int_\Omega |f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) \right. \\ & \quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^2 |D^\beta v|^2 \right)^{1/2}. \end{aligned}$$

Note that for fixed  $u, v \in W^{m,p}(\Omega)$  and all  $s, t \in [0, 1]$  the functions

$$|f_{\alpha\beta}(x, u(x) + stv(x), \dots, D^m u(x) + stD^m v(x)) - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^2$$

are uniformly bounded. It follows

$$\lim_{t \rightarrow 0} \| [D_\alpha \mathcal{F}(u + tv) - D_\alpha \mathcal{F}(u)]/t - A_\alpha(u)v \| = 0.$$

Hence  $D_\alpha \mathcal{F}$  has Gâteaux derivative  $A_\alpha(u)$  at  $u$ .

**Step 3.** *Prove that  $D\mathcal{F}'$  is bounded on any ball in  $W^{m,p}(\Omega)$ .* From the above arguments we obtain  $C = C(m, n, p, \Omega)$  such that

$$\begin{aligned} |\langle A_\alpha(u)v, \varphi \rangle| &= \left| \int_\Omega f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v D^\alpha \varphi(x) dx \right| \\ &\leq C \|v\|_{m,p} \cdot \|\varphi\|_{m,p} \left( \int_\Omega |f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \end{aligned} \quad (4.24)$$

for  $|\alpha| + |\beta| < 2m$  or  $p > 2$  and  $|\alpha| = |\beta| = m$ , and

$$\begin{aligned} |\langle A_\alpha(u)v, \varphi \rangle| &= \left| \int_\Omega f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v D^\alpha \varphi(x) dx \right| \\ &\leq C \cdot \|\varphi\|_{m,p} \left( \int_\Omega |f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^2 |D^\beta v|^2 \right)^{1/2} \end{aligned} \quad (4.25)$$

for  $p = 2$  and  $|\alpha| = |\beta| = m$ .

For (4.25), by (1.5) we derive

$$|f_{\alpha\beta}(x, u(x), \dots, D^m u(x))| \leq \sup_{k < m-n/2} \mathfrak{g}_1(\|u\|_{C^k}) \quad \forall u \in W^{m,2},$$

and hence

$$\begin{aligned} |\langle A_\alpha(u)v, \varphi \rangle| &= \left| \int_\Omega f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v dx \right| \\ &\leq C \sup_{k < m-n/2} \mathfrak{g}_1(\|u\|_{C^k}) \left( \int_\Omega |D^\beta v|^2 \right)^{1/2} \end{aligned}$$

This and (4.24), (1.5) yield the desired claim.

**Step 4.** *Prove (ii) in B).* By (4.2) we derive

$$\begin{aligned} & \langle \varphi, [D\mathcal{F}'(u_k)]^* v - [D\mathcal{F}'(u)]^* v \rangle = \langle D\mathcal{F}'(u_k)\varphi - D\mathcal{F}'(u)\varphi, v \rangle \\ &= \sum_{|\alpha|, |\beta| \leq m} \int_\Omega [f_{\alpha\beta}(x, u_k(x), \dots, D^m u_k(x)) \\ & \quad - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))] D^\beta \varphi \cdot D^\alpha v dx. \end{aligned}$$



Let  $u_k \rightarrow u$  in  $W^{m,p}(\Omega)$ . If  $p > 2$ , with the same reasoning as in (4.24) we can derive from this that

$$\begin{aligned}
& |\langle \varphi, [D\mathcal{F}'(u_k)]^* v - [D\mathcal{F}'(u)]^* v \rangle| \\
& \leq \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} |f_{\alpha\beta}(x, u_k(x), \dots, D^m u_k(x)) \\
& \quad - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))| \cdot |D^{\beta} \varphi| \cdot |D^{\alpha} v| dx \\
& \leq C \|v\|_{m,p} \|\varphi\|_{m,p} \sum_{|\alpha|, |\beta| \leq m} \left( \int_{\Omega} |f_{\alpha\beta}(x, u_k(x), \dots, D^m u_k(x)) \right. \\
& \quad \left. - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} dx \right)^{p_{\alpha\beta}}.
\end{aligned}$$

Because of (1.5), as treated in Step 1 of Section 4.2 we deduce

$$\sum_{|\alpha|, |\beta| \leq m} \left( \int_{\Omega} |f_{\alpha\beta}(x, u_k(x), \dots, D^m u_k(x)) - f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} dx \right)^{p_{\alpha\beta}} \rightarrow 0$$

and so  $\|[D\mathcal{F}'(u_k)]^* v - [D\mathcal{F}'(u)]^* v\| \rightarrow 0$  as  $k \rightarrow \infty$ .

For  $p = 2$  we can use (4.25) to arrive at the same conclusion.

**Step 5.** The proof of (iii) in **B**) is the same as that of (iii) in **D**) later on.

#### 4.4 Proof for C) of Theorem 4.1

Let  $u_j \rightarrow u$  in  $W^{m,p}(\Omega)$  and satisfy  $\overline{\lim}_{j \rightarrow \infty} \langle \mathcal{F}'(u_j), u_j - u \rangle \leq 0$ , i.e.,

$$\overline{\lim}_{j \rightarrow \infty} \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha}(x, u_j(x), \dots, D^m u_j(x)) D^{\alpha}(u_j - u) dx = 0. \quad (4.26)$$

By Sobolev embedding theorem we have strong convergence

$$\begin{aligned}
D^{\alpha} u_j & \rightarrow D^{\alpha} u \quad \text{in } L^q(\Omega) \text{ if } q < \frac{np}{n-p}, \quad m - n/p \leq |\alpha| \leq m-1, \\
D^{\alpha} u_j & \rightarrow D^{\alpha} u \quad \text{in } C^0(\Omega) \text{ if } |\alpha| < m - n/p.
\end{aligned}$$

Since  $\sup_j \|u_j\|_{m,p} < \infty$ ,  $C = \sup\{\|u\|_{C^k} + \|u_j\|_{C^k} : k < m - n/p, j \in \mathbb{N}\} < \infty$ . These, (4.10) and (4.11) lead to

$$\begin{aligned}
& \sum_{|\alpha| < m-n/p} \int_{\Omega} |f_{\alpha}(x, u_j(x), \dots, D^m u_j(x)) D^{\alpha}(u_j - u)| dx \\
& \leq \sum_{|\alpha| < m-n/p} \int_{\Omega} |f_{\alpha}(x, 0)| \cdot |D^{\alpha}(u_j - u)| dx + \mathfrak{g}_5(C) \sum_{|\alpha| < m-n/p} \int_{\Omega} |D^{\alpha}(u_j - u)| dx \\
& \quad + \mathfrak{g}_5(C) \sum_{|\alpha| < m-n/p} \sum_{m-n/p \leq |\gamma| \leq m} \int_{\Omega} |D^{\gamma} u_j|^{p_{\gamma}} \cdot |D^{\alpha}(u_j - u)| dx \rightarrow 0
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{m-n/p \leq |\alpha| \leq m-1} \int_{\Omega} |f_{\alpha}(x, u_j(x), \dots, D^m u_j(x)) D^{\alpha}(u_j - u)| dx \\
& \leq \sum_{m-n/p \leq |\alpha| \leq m-1} \int_{\Omega} |f_{\alpha}(x, 0)| \cdot |D^{\alpha}(u_j - u)| dx + \mathfrak{g}_5(C) \sum_{m-n/p \leq |\alpha| \leq m-1} \int_{\Omega} |D^{\alpha}(u_j - u)| dx \\
& \quad + \mathfrak{g}_5(C) \sum_{m-n/p \leq |\alpha| \leq m-1} \int_{\Omega} \left( \sum_{m-n/p \leq |\gamma| \leq m} |D^{\gamma} u_j|^{p_{\gamma}} \right)^{1/q_{\alpha}} \cdot |D^{\alpha}(u_j - u)| dx \rightarrow 0.
\end{aligned}$$

Here the final limit is because Lemma 4.5(ii) implies that

$$\begin{aligned}
& \int_{\Omega} \left( \sum_{m-n/p \leq |\gamma| \leq m} |D^{\gamma} u_j|^{p_{\gamma}} \right)^{1/q_{\alpha}} \cdot |D^{\alpha}(u_j - u)| dx \rightarrow 0 \\
& \leq \sum_{m-n/p \leq |\gamma| \leq m} \int_{\Omega} |D^{\gamma} u_j|^{p_{\gamma}/q_{\alpha}} \cdot |D^{\alpha}(u_j - u)| dx \rightarrow 0 \\
& \leq \sum_{m-n/p \leq |\gamma| \leq m} \left( \int_{\Omega} |D^{\gamma} u_j|^{p_{\gamma}} dx \right)^{1/q_{\alpha}} \left( \int_{\Omega} |D^{\alpha}(u_j - u)|^{p_{\alpha}} dx \right)^{1/p_{\alpha}} \rightarrow 0.
\end{aligned}$$

It follows that

$$\sum_{|\alpha| \leq m-1} \int_{\Omega} |f_{\alpha}(x, u_j(x), \dots, D^m u_j(x)) D^{\alpha}(u_j - u)| dx \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (4.27)$$

This and (4.26) yield

$$\sum_{|\alpha|=m} \int_{\Omega} f_{\alpha}(x, u_j(x), \dots, D^m u_j(x)) D^{\alpha}(u_j - u) dx \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (4.28)$$

Next, we claim

$$\begin{aligned}
& \sum_{|\alpha|=m} \int_{\Omega} |f_{\alpha}(x, u_j(x), \dots, D^{m-1} u_j(x), D^m u(x)) D^{\alpha}(u_j - u)| dx \\
& \leq \sum_{|\alpha|=m} \int_{\Omega} \left| [f_{\alpha}(x, u_j(x), \dots, D^{m-1} u_j(x), D^m u(x)) \right. \\
& \quad \left. - f_{\alpha}(x, u(x), \dots, D^{m-1} u(x), D^m u(x))] D^{\alpha}(u_j - u) \right| dx \\
& \quad + \sum_{|\alpha|=m} \int_{\Omega} |f_{\alpha}(x, u(x), \dots, D^{m-1} u(x), D^m u(x)) D^{\alpha}(u_j - u)| dx \\
& = I_{1,j} + I_{2,j} \rightarrow 0.
\end{aligned} \quad (4.29)$$

Indeed, by (4.11) it is easily checked that  $f_{\alpha}(x, u(x), \dots, D^{m-1} u(x), D^m u(x))$  belongs to  $L^{q_{\alpha}}(\Omega)$  for  $|\alpha| \geq m - n/p$ . Since  $u_j \rightharpoonup u$  in  $W^{m,p}(\Omega)$  it follows from that  $I_{2,j} \rightarrow 0$  as  $j \rightarrow \infty$ .

By the Hölder inequality we have

$$I_{1,j} \leq \sum_{|\alpha|=m} \left( \int_{\Omega} |f_{\alpha}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u(x)) - f_{\alpha}(x, u(x), \dots, D^{m-1}u(x), D^m u(x))|^{q_{\alpha}} dx \right)^{1/q_{\alpha}} \left( \int_{\Omega} |D^{\alpha}(u_j - u)|^{p_{\alpha}} dx \right)^{1/p_{\alpha}}.$$

As before, using (4.11) and Proposition 4.4 it is easy to derive

$$\left( \int_{\Omega} |f_{\alpha}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u(x)) - f_{\alpha}(x, u(x), \dots, D^{m-1}u(x), D^m u(x))|^{q_{\alpha}} dx \right)^{1/q_{\alpha}} \rightarrow 0.$$

Note that the sequence  $\left( \int_{\Omega} |D^{\alpha}(u_j - u)|^{p_{\alpha}} dx \right)^{1/p_{\alpha}}$  is bounded. We get  $I_{1,j} \rightarrow 0$ .

Hence (4.28) and (4.29) yield

$$\sum_{|\alpha|=m} \int_{\Omega} [f_{\alpha}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u_j(x)) - f_{\alpha}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u(x))] D^{\alpha}(u_j - u) dx \rightarrow 0. \quad (4.30)$$

Note that the integrand in each term is non-negative, which may be derived from the mean value theorem and (1.6) as seen below. This implies that for any subset  $E \subset \Omega$ ,

$$\lambda_j(E) := \int_E \sum_{|\alpha|=m} [f_{\alpha}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u_j(x)) - f_{\alpha}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u(x))] D^{\alpha}(u_j - u) dx \rightarrow 0 \quad (4.31)$$

as  $j \rightarrow \infty$ . We claim that for  $E \subset \Omega$ ,

$$\lim_{\text{meas}(E) \rightarrow 0} \int_E \sum_{|\alpha|=m} |D^{\alpha}u_j(x)|^p dx = 0 \quad (4.32)$$

uniformly with respect to  $j$ . In fact, by (4.31) we have with  $v_j = u_j - u$ ,

$$\lambda_j(E) = \sum_{|\alpha|=|\beta|=m} \int_E \int_0^1 f_{\alpha\beta}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u(x) + s D^m v_j(x)) D^{\beta}v_j \cdot D^{\alpha}v_j ds dx.$$

From (1.6) and Lemma 4.5(i) we deduce

$$\begin{aligned}
& \sum_{|\alpha|=|\beta|=m} \int_E \int_0^1 f_{\alpha\beta}(x, u_j(x), \dots, D^{m-1}u_j(x), D^m u(x) + sD^m v_j(x)) D^\beta v_j \cdot D^\alpha v_j ds dx \\
& \geq \int_E \int_0^1 \mathfrak{g}_2 \left( \sum_{k < m-n/p} \|u_j\|_{C^k} \right) \left( 1 + \sum_{|\gamma|=m} |D^\gamma u(x) + sD^\gamma v_j(x)| \right)^{p-2} \sum_{|\alpha|=m} |D^\alpha v_j|^2 ds dx \\
& \geq \mathfrak{g}_2 \left( \sum_{k < m-n/p} \|u_j\|_{C^k} \right) \int_E \int_0^1 (1 + |D^\gamma u(x) + sD^\gamma v_j(x)|)^{p-2} \sum_{|\alpha|=m} |D^\alpha v_j|^2 ds dx \\
& \geq \mathfrak{g}_2 \left( \sum_{k < m-n/p} \|u_j\|_{C^k} \right) \int_E (1 + |D^\gamma u(x)| + |D^\gamma u_j(x)|)^{p-2} \sum_{|\alpha|=m} |D^\alpha v_j|^2 dx \\
& \geq \mathfrak{g}_2 \left( \sum_{k < m-n/p} \|u_j\|_{C^k} \right) \int_E (1 + |D^\gamma v_j(x)|)^{p-2} |D^\gamma v_j|^2 dx \\
& \geq \mathfrak{g}_2 \left( \sum_{k < m-n/p} \|u_j\|_{C^k} \right) \int_E |D^\gamma v_j|^p dx
\end{aligned} \tag{4.33}$$

for each  $\gamma$  of length  $m$ . It follows

$$\begin{aligned}
\sum_{|\gamma|=m} \int_E |D^\gamma v_j|^p dx & \leq \frac{M_0(m)}{\mathfrak{g}_2(\sum_{k < m-n/p} \|u_j\|_{C^k})} \lambda_j(E) \\
& \leq \frac{M_0(m)}{\mathfrak{g}_2(\sup_j \sum_{k < m-n/p} \|u_j\|_{C^k})} \lambda_j(E).
\end{aligned} \tag{4.34}$$

The final inequality is because  $u_j \rightharpoonup u$  and thus that  $\sup_j \|u_j\|_{m,p} < \infty$ , which implies that  $\sup_j \sum_{k < m-n/p} \|u_j\|_{C^k} < \infty$ . Moreover

$$\left( \sum_{|\gamma|=m} \int_E |D^\gamma u_j|^p dx \right)^{1/p} \leq \left( \sum_{|\gamma|=m} \int_E |D^\gamma u|^p dx \right)^{1/p} + \left( \sum_{|\gamma|=m} \int_E |D^\gamma v_j|^p dx \right)^{1/p}.$$

Given any  $\varepsilon > 0$ , from (4.31) and (4.34) there exist  $j_0 \in \mathbb{N}$  such that

$$\left( \sum_{|\gamma|=m} \int_E |D^\gamma v_j|^p dx \right)^{1/p} < \varepsilon^p / 2$$

for all  $j \geq j_0$  and all  $E \subset \Omega$ . Using the absolute continuity of the integral we have  $\delta > 0$  such that for any  $E \subset \Omega$  with  $\text{mes}(E) < \delta$ ,

$$\left( \sum_{|\gamma|=m} \int_E |D^\gamma u|^p dx \right)^{1/p} + \left( \sum_{|\gamma|=m} \int_E |D^\gamma v_j|^p dx \right)^{1/p} < \varepsilon^p / 2, \quad j = 1, \dots, j_0.$$

These lead to

$$\sum_{|\gamma|=m} \int_E |D^\gamma u_j|^p dx < \varepsilon$$

for all  $j \in \mathbb{N}$  and all  $E \subset \Omega$  with  $\text{mes}(E) < \delta$ . (4.32) is proved.

Next, by (4.34) and (4.31), for any given  $\sigma > 0$  we have

$$\begin{aligned} \sigma^p \text{mes}(\{|D^\alpha v_j| \geq \sigma\}) &\leq \int_{\{|D^\alpha v_j| \geq \sigma\}} |D^\alpha v_j|^p dx \leq \int_{\Omega} |D^\alpha v_j|^p dx \\ &\leq \frac{M_0(m)}{\mathfrak{g}_2(\sup_j \sum_{k < m-n/p} \|u_j\|_{C^k})} \lambda_j(\Omega) \rightarrow 0 \end{aligned}$$

as  $j \rightarrow \infty$ . This means that the sequence  $D^\alpha u_j$  converges to  $D^\alpha u$  in measure for  $|\alpha| = m$ . Combing with (4.32) we obtain that  $D^\alpha u_j \rightarrow D^\alpha u$  in  $L^p(\Omega)$  for  $|\alpha| = m$ . Moreover,  $u_j \rightarrow u$  implies that  $u_j \rightarrow u$  in  $W^{m-1,p}(\Omega)$ . Hence  $\|u_j - u\|_{m,p} \rightarrow 0$  as  $j \rightarrow \infty$ .  $\square$

#### 4.5 Proof for D) of Theorem 4.1

**Step 1.** (i) of **D)** can be proved as in (ii) of **B)**.

**Step 2.** *Prove (ii) of D).* By Sobolev embedding theorem

$$\sup\{\|D^\gamma u\|_{C^k} : k < m - n/2, u \in W^{m,2}(\Omega), \|u\|_{m,2} \leq R\} \in (0, \infty).$$

Let  $C$  be equal to the value of  $\mathfrak{g}_2$  at this number. We derive from (1.6) that

$$\begin{aligned} (P(u)v, v)_{m,2} &= \sum_{|\alpha|=|\beta|=m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^\beta v \cdot D^\alpha v dx \\ &\quad + \sum_{|\alpha| \leq m-1} \int_{\Omega} D^\alpha v \cdot D^\alpha v dx \\ &\geq C \sum_{|\alpha|=m} \int_{\Omega} |D^\alpha v|^2 dx + \sum_{|\alpha| \leq m-1} \int_{\Omega} D^\alpha v \cdot D^\alpha v dx \\ &\geq \min\{C, 1\} \|v\|_{m,2}^2 \quad \text{for any } v \in W^{m,2}(\Omega). \end{aligned}$$

**Step 3.** *Prove (iii) of D).* As in the proof of (4.24) we can obtain  $C = C(m, n, p, \Omega) > 0$  with  $p = 2$  such that

$$\begin{aligned} &([Q(u) - Q(\bar{u})]v, \varphi)_{m,2} \\ &= \left| \sum_{|\alpha|+|\beta| < 2m} \int_{\Omega} [f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) - f_{\alpha\beta}(x, \bar{u}(x), \dots, D^m \bar{u}(x))] D^\beta v D^\alpha \varphi(x) dx \right| \\ &\leq C \sum_{|\alpha|+|\beta| < 2m} \left( \int_{\Omega} |f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) - f_{\alpha\beta}(x, \bar{u}(x), \dots, D^m \bar{u}(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \times \\ &\quad \times \|v\|_{m,p} \cdot \|\varphi\|_{m,p}. \end{aligned}$$

So it follows from (1.5) and Proposition 4.4 that

$$\begin{aligned} &\|Q(u) - Q(\bar{u})\|_{\mathcal{L}(W^{m,2}(\Omega))} \\ &\leq C \sum_{|\alpha|+|\beta| < 2m} \left( \int_{\Omega} |f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) - f_{\alpha\beta}(x, \bar{u}(x), \dots, D^m \bar{u}(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \rightarrow 0. \end{aligned}$$

To prove the second claim let us decompose  $Q(u)$  into  $Q_1(u) + Q_2(u) + Q_3(u)$ , where

$$\begin{aligned}
(Q_1(u)v, \varphi)_{m,2} &= \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx \\
&\quad - \sum_{|\alpha| \leq m-1} \int_{\Omega} D^{\alpha} v \cdot D^{\alpha} \varphi dx, \\
(Q_2(u)v, \varphi)_{m,2} &= \sum_{|\alpha|=m, |\beta| \leq m-1} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx, \\
(Q_3(u)v, \varphi)_{m,2} &= \sum_{|\alpha| \leq m-1, |\beta|=m} \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta} v \cdot D^{\alpha} \varphi dx.
\end{aligned}$$

Clearly,  $(Q_2(u)v, \varphi)_{m,2} = (Q_3(u)\varphi, v)_{m,2}$ , that is, they are adjoint each other. Let  $v_j \rightharpoonup v$  in  $W^{m,2}(\Omega)$ . By the proof of (4.24) we can get  $C = C(m, n, p, \Omega) > 0$  with  $p = 2$  such that

$$\begin{aligned}
& |(Q_1(u)(v_j - v), \varphi)_{m,2}| \\
& \leq \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \left| \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta}(v_j - v) D^{\alpha} \varphi(x) dx \right| \\
& \quad + \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^{\alpha}(v_j - v)| \cdot |D^{\alpha} \varphi| dx \\
& \leq C \|v_j - v\|_{m-1,2} \cdot \|\varphi\|_{m-1,2} \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \left( \int_{\Omega} |f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \\
& \quad + \|v_j - v\|_{m-1,2} \cdot \|\varphi\|_{m-1,2}
\end{aligned}$$

and hence

$$\begin{aligned}
& \|Q_1(u)(v_j - v)\|_{m,2} \\
& \leq C \|v_j - v\|_{m-1,2} \sum_{|\alpha| \leq m-1, |\beta| \leq m-1} \left( \int_{\Omega} |f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \\
& \quad + \|v_j - v\|_{m-1,2} \rightarrow 0
\end{aligned}$$

because  $\|v_j - v\|_{m-1,2} \rightarrow 0$  by the compactness of the embedding  $W^{m,2}(\Omega) \hookrightarrow W^{m-1,2}(\Omega)$ .

Similarly, we have

$$\begin{aligned}
& |(Q_2(u)(v_j - v), \varphi)_{m,2}| \\
& \leq \sum_{|\alpha|=m, |\beta| \leq m-1} \left| \int_{\Omega} f_{\alpha\beta}(x, u(x), \dots, D^m u(x)) D^{\beta}(v_j - v) D^{\alpha} \varphi(x) dx \right| \\
& \leq C \|v_j - v\|_{m-1,2} \cdot \|\varphi\|_{m,2} \sum_{|\alpha|=m, |\beta| \leq m-1} \left( \int_{\Omega} |f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}}
\end{aligned}$$

and hence

$$\begin{aligned}
& \|Q_2(u)(v_j - v)\|_{m,2} \\
& \leq C \|v_j - v\|_{m-1,2} \sum_{|\alpha|=m, |\beta| \leq m-1} \left( \int_{\Omega} |f_{\alpha\beta}(x, u(x), \dots, D^m u(x))|^{1/p_{\alpha\beta}} \right)^{p_{\alpha\beta}} \rightarrow 0.
\end{aligned}$$

Since  $Q_3(u)$  is the adjoint operator of  $Q_2(u)$ , it is completely continuous too.

**Step 4.** *Prove (iv) of D).* By the arguments in Step 3 we see: for every given  $R > 0$  there exist positive constants  $\hat{C}(R, n, m, \Omega)$  such that if  $u \in W^{m,2}(\Omega)$  satisfies  $\|u\|_{m,2} \leq R$  then

$$\begin{aligned} |(Q_1(u)v, \varphi)_{m,2}| &\leq \hat{C}\|v\|_{m-1,2} \cdot \|\varphi\|_{m-1,2} & \forall v, \varphi \in W^{m,2}(\Omega), \\ |(Q_2(u)v, \varphi)_{m,2}| &\leq \hat{C}\|v\|_{m-1,2} \cdot \|\varphi\|_{m,2} & \forall v, \varphi \in W^{m,2}(\Omega), \\ |(Q_3(u)v, \varphi)_{m,2}| &\leq \hat{C}\|v\|_{m,2} \cdot \|\varphi\|_{m-1,2} & \forall v, \varphi \in W^{m,2}(\Omega) \end{aligned}$$

and therefore

$$|(Q(u)v, v)_{m,2}| \leq 3\hat{C}\|v\|_{m-1,2} \cdot \|v\|_{m,2} \quad \forall v \in W^{m,2}(\Omega).$$

For the constant  $C$  in (ii) of **D)**, using the inequality  $ab \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$  for any  $\varepsilon > 0$  and  $a, b \geq 0$ , we derive with  $\varepsilon = C/(3\hat{C})$ ,

$$|(Q(u)v, v)_{m,2}| \leq 3\hat{C}\|v\|_{m-1,2} \cdot \|v\|_{m,2} \leq \frac{C}{2}\|v\|_{m,2}^2 + \frac{9\hat{C}^2}{2C}\|v\|_{m-1,2}^2$$

for all  $v \in W^{m,2}(\Omega)$ . Taking  $C_1 = C/2$  and  $C_2 = \frac{9\hat{C}^2}{2C}$ , this and (ii) of **D)** give the desired result.

## 5 (PS)- and (C)-conditions

A  $C^1$  functional  $\varphi$  on a Banach-Finsler manifold  $\mathcal{M}$  is said to satisfy the *Palais-Smale condition at the level*  $c \in \mathbb{R}$  ( $(PS)_c$ -condition, for short) if every sequence  $\{x_j\}_{j \geq 1} \subset X$  such that  $\varphi(x_j) \rightarrow c \in \mathbb{R}$  and  $\varphi'(x_j) \rightarrow 0$  in  $X^*$  has a convergent subsequence in  $\mathcal{M}$ . When  $\varphi$  satisfies the  $(PS)_c$ -condition at every level  $c \in \mathbb{R}$  we say that it satisfies the *Palais-Smale condition* ( $(PS)$ -condition, for short).

When  $\mathcal{M}$  a Banach space there is weaker condition. Call a  $C^1$  functional  $\varphi$  on a Banach space  $X$  to satisfy the *Cerami condition at the level*  $c \in \mathbb{R}$  ( $(C)_c$ -condition, for short) if every sequence  $\{x_j\}_{j \geq 1} \subset X$  such that  $\varphi(x_j) \rightarrow c \in \mathbb{R}$  and  $(1 + \|x_j\|)\varphi'(x_j) \rightarrow 0$  in  $X^*$  has a convergent subsequence in  $X$ . When  $\varphi$  satisfies the  $(C)_c$ -condition at every level  $c \in \mathbb{R}$  we say that it satisfies the *Cerami condition* ( $(C)$ -condition, for short).

Actually, if a  $C^1$  functional  $\varphi$  on a Banach space  $X$  is bounded below, Caklovic, Li and Willem [10] showed that  $\varphi$  satisfies the  $(PS)$ -condition if and only if it is coercive. It was further proved in [52, Proposition 5.23] that  $\varphi$  satisfies the  $(PS)$ -condition if and only if it does the  $(C)$ -condition. Recently, it was proved in [62, Theorem 6] that if a continuous functional  $\varphi$  on  $X$  is Gâteaux differentiable and satisfies the weak Palais-Smale condition then  $|\varphi|$  is coercive provided  $\{x \in X \mid \varphi(x) = c\}$  is bounded for some  $c \in \mathbb{R}$ .

**Theorem 5.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $N \in \mathbb{N}$ ,  $p \in [2, \infty)$  and  $V \subset W^{m,p}(\Omega, \mathbb{R}^N)$  be as in Theorem 4.2. Suppose that Hypothesis  $\mathfrak{F}_{p,N}$  hold and that  $\mathfrak{F}$  is coercive, i.e.,  $\mathfrak{F}(\vec{u}) \rightarrow \infty$  as  $\|\vec{u}\|_V \rightarrow \infty$ . Then  $\mathfrak{F}$  satisfies the (PS)- and (C)-conditions on  $V$ . In particular, the same conclusions hold with the functional  $\mathcal{F}$  on  $V \subset W^{m,p}(\Omega)$  under the condition  $\mathfrak{f}_p$ .*

*Proof.* Since  $\mathfrak{F}$  is coercive, it is bounded below. By [52, Proposition 5.23] it suffices to prove that  $\mathfrak{F}$  satisfies the (PS)-condition.

Let a sequence  $\{\vec{u}_j\}_{j \geq 1}$  such that  $\mathfrak{F}(\vec{u}_j) \rightarrow c \in \mathbb{R}$  and  $\mathfrak{F}'(\vec{u}_j) \rightarrow 0$  as  $j \rightarrow \infty$ . Since  $\mathfrak{F}$  is coercive, the sequence  $\{\vec{u}_j\}_{j \geq 1}$  must be bounded. Note that  $V$  is a self-reflexive Banach space. After passing to a subsequence we may assume  $\vec{u}_j \rightharpoonup \vec{u}$  in  $V$ . Moreover,  $\mathfrak{F}'(\vec{u}_j) \rightarrow 0$  implies  $\overline{\lim}_{j \rightarrow \infty} \langle \mathfrak{F}'(\vec{u}_j), \vec{u}_j - \vec{u} \rangle = 0$ . By Theorem 4.2 (the corresponding conclusion to **C**) of Theorem 4.1) we know that  $\mathfrak{F}'$  is of class  $(S)_+$ . Hence  $\vec{u}_j \rightarrow \vec{u}$  in  $V$ .  $\square$

Clearly, the coercivity of  $\mathfrak{F}$  implies that it is bounded below. On the other hand, for a  $C^1$  functional  $\varphi$  on a Banach space  $X$  which is bounded below, Li Shujie showed that it is coercive if  $\varphi$  satisfies the (PS)-condition.

There exist some explicit conditions on  $F$  under which  $\mathfrak{F}$  is coercive on  $W_0^{m,p}(\Omega, \mathbb{R}^N)$ , for example, there exist some two positive constants  $c_0, c_1$  such that

$$F(x, \xi) \geq c_0 \sum_{i=1}^N \sum_{|\alpha|=m} |\xi_\alpha^i|^p - c_1 \quad \forall (x, \xi).$$

The coercivity requirement is too strong. In fact, the proof of Theorem 5.1 shows that under Hypothesis  $\mathfrak{F}_{p,N}$  we only need to add some conditions so that

$$\sup_j |\mathfrak{F}(\vec{u}_j)| < \infty \quad \text{and} \quad \mathfrak{F}'(\vec{u}_j) \rightarrow 0 \implies \sup_j \|\vec{u}_j\|_{m,p} < \infty.$$

**Theorem 5.2.** *Under Hypothesis  $\mathfrak{F}_{p,N}$ , suppose that there exist  $\kappa \in \mathbb{R}$  and  $\Upsilon \in L^1(\Omega)$  such that*

$$F(x, \xi) - \kappa \sum_{i=1}^N \sum_{|\alpha| \leq m} F_\alpha^i(x, \xi) \xi_\alpha^i \geq c_0 \sum_{i=1}^N \sum_{|\alpha|=m} |\xi_\alpha^i|^p - c_1 \sum_{i=1}^N |\xi_0^i|^p - \Upsilon(x) \quad \forall (x, \xi),$$

where  $c_0 > 0$  and  $c_0 - c_1 S_{m,p} > 0$  for the best constant  $S_{m,p} > 0$  with

$$\int_\Omega |u|^p dx \leq S_{m,p} \int_\Omega |D^m u|^p dx = S_{m,p} \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p \quad \forall u \in W_0^{m,p}(\Omega).$$

Then  $\mathfrak{F}$  satisfies the (PS)- and (C)-conditions on  $W_0^{m,p}(\Omega, \mathbb{R}^N)$ .

*Proof.* Let  $\{\vec{u}_k\}_{k \geq 1} \subset W_0^{m,p}(\Omega, \mathbb{R}^N)$  be a sequence such that  $|\mathfrak{F}(\vec{u}_k)| \leq M \forall k$  for some  $M > 0$ , and  $\mathfrak{F}'(\vec{u}_k) \rightarrow 0$  (resp.  $(1 + \|\vec{u}_k\|)\mathfrak{F}'(\vec{u}_k) \rightarrow 0$ ). By (4.7) the latter means

$$\left| \sum_{i=1}^N \sum_{|\alpha| \leq m} \int_\Omega F_\alpha^i(x, \vec{u}_k(x), \dots, D^m \vec{u}_k(x)) D^\alpha u_k^i dx \right| \leq \varepsilon_k \|\vec{u}_k\|_{m,p} \quad (5.1)$$

(resp.  $\left| \sum_{i=1}^N \sum_{|\alpha| \leq m} \int_\Omega F_\alpha^i(x, \vec{u}_k(x), \dots, D^m \vec{u}_k(x)) D^\alpha u_k^i dx \right| \leq \varepsilon_k \frac{\|\vec{u}_k\|_{m,p}}{1 + \|\vec{u}_k\|_{m,p}} )$



where  $\varepsilon_k \rightarrow 0$  and  $\|\vec{u}_k\|_{m,p} = \|D^m \vec{u}_k\|_p$  as usual. By the assumption we have

$$\begin{aligned} & \mathfrak{F}(\vec{u}_k) - \kappa \sum_{i=1}^N \sum_{|\alpha| \leq m} \int_{\Omega} F_{\alpha}^i(x, \vec{u}_k(x), \dots, D^m \vec{u}_k(x)) D^{\alpha} u_k^i dx \\ & \geq c_0 \sum_{i=1}^N \sum_{|\alpha|=m} \int_{\Omega} |D^{\alpha} u_k^i|^p dx - c_1 \sum_{i=1}^N \int_{\Omega} |u_k^i|^p dx - \int_{\Omega} \Upsilon(x) dx \\ & \geq c_0 \sum_{i=1}^N \int_{\Omega} |D^m u_k^i|^p dx - c_1 S_{m,p} \sum_{i=1}^N \int_{\Omega} |D^m u_k^i|^p dx - \int_{\Omega} \Upsilon(x) dx \end{aligned}$$

and therefore

$$(c_0 - c_1 S_{m,p}) \sum_{i=1}^N \int_{\Omega} |D^m u_k^i|^p dx \leq \int_{\Omega} \Upsilon(x) dx + M + |\kappa| \varepsilon_k \|\vec{u}_k\|_{m,p}$$

$$(\text{resp. } (c_0 - c_1 S_{m,p}) \sum_{i=1}^N \int_{\Omega} |D^m u_k^i|^p dx \leq \int_{\Omega} \Upsilon(x) dx + M + |\kappa| \varepsilon_k).$$

This implies that  $\|\vec{u}_k\|_{m,p}$  is bounded. Passing to a subsequence if necessary, we may assume  $\vec{u}_k \rightharpoonup \vec{u}$ . The remainder is the same as that of Theorem 5.1.  $\square$

**Theorem 5.3.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Sobolev domain. Suppose that **Hypothesis**  $\mathfrak{F}_{2,N}$  is satisfied with the constant function  $\mathfrak{g}_2$ , and that for any  $(x, \xi) \in \overline{\Omega} \times \prod_{k=0}^{m-1} \mathbb{R}^{N \times M_0(k)}$ ,*

$$F(x, \xi, \mathbf{0}) \leq \varphi(x) + C \sum_{i=1}^N \sum_{|\alpha| \leq m-1} |\xi_{\alpha}^i|^r,$$

where  $\varphi \in L^1(\Omega)$  and  $1 \leq r < 2$ . Then  $\mathfrak{F}$  satisfies the (PS)- and (C)-conditions on  $W_0^{m,2}(\Omega, \mathbb{R}^N)$ .

*Proof.* For any  $x \in \Omega$  and  $\xi = (\xi^1, \dots, \xi^m) \in \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)}$  let  $\hat{\xi} = (\xi^1, \dots, \xi^{m-1}) \in \prod_{k=0}^{m-1} \mathbb{R}^{N \times M_0(k)}$ . By the mean value theorem we get

$$\begin{aligned} & \sum_{i=1}^N \sum_{|\alpha|=m} F_{\alpha}^i(x, \xi) \xi_{\alpha}^i - F(x, \xi) \\ &= \sum_{i=1}^N \sum_{|\alpha|=m} F_{\alpha}^i(x, \xi) \xi_{\alpha}^i - [F(x, \xi) - F(x, \hat{\xi}, \mathbf{0})] - F(x, \hat{\xi}, \mathbf{0}) \\ &= \sum_{i=1}^N \sum_{|\alpha|=m} F_{\alpha}^i(x, \xi) \xi_{\alpha}^i - \sum_{i=1}^N \sum_{|\alpha|=m} \int_0^1 F_{\alpha}^i(x, \hat{\xi}, t \xi^m) \xi_{\alpha}^i dt - F(x, \hat{\xi}, \mathbf{0}) \\ &= \sum_{i,j=1}^N \sum_{|\alpha|=|\beta|=m} \int_0^1 dt \int_0^1 F_{\alpha\beta}^{ij}(x, \hat{\xi}, (t+s-st)\xi^m) (1-t) \xi_{\alpha}^i \xi_{\beta}^j ds - F(x, \hat{\xi}, \mathbf{0}) \\ &\geq \frac{1}{2} \mathfrak{g}_2 \sum_{i=1}^N \sum_{|\alpha|=m} |\xi_{\alpha}^i|^2 - F(x, \hat{\xi}, \mathbf{0}). \end{aligned}$$

It follows that for any  $\vec{u} \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ ,

$$\frac{1}{2} \mathfrak{g}_2 \sum_{i=1}^N \sum_{|\alpha|=m} \int_{\Omega} |D^{\alpha} u^i|^2 \leq \int_{\Omega} F(x, \vec{u}, \dots, D^{m-1} \vec{u}, 0) - \mathfrak{F}(\vec{u}) + \langle \mathfrak{F}'(\vec{u}), \vec{u} \rangle.$$

By the assumption and the Young inequality we derive

$$\begin{aligned} \int_{\Omega} F(x, \vec{u}, \dots, D^{m-1} \vec{u}, 0) &\leq \int_{\Omega} \varphi(x) + C \sum_{i=1}^N \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^{\alpha} u^i|^r \\ &\leq \int_{\Omega} \varphi(x) + C \sum_{i=1}^N \sum_{|\alpha| \leq m-1} \int_{\Omega} \left( \frac{r\varepsilon}{2} |D^{\alpha} u^i|^2 + \frac{2-r}{2} \varepsilon^{-r/(2-r)} \right) \\ &\leq \int_{\Omega} \varphi(x) + C\varepsilon \|\vec{u}\|_{m,2}^2 + C\varepsilon^{-r/(2-r)} \end{aligned}$$

and hence

$$\frac{1}{2} \mathfrak{g}_2 \|\vec{u}\|_{m,2}^2 \leq \int_{\Omega} \varphi(x) + C\varepsilon \|\vec{u}\|_{m,2}^2 + C\varepsilon^{-r/(2-r)} - \mathfrak{F}(\vec{u}) + \langle \mathfrak{F}'(\vec{u}), \vec{u} \rangle.$$

Taking  $\varepsilon = \frac{\mathfrak{g}_2}{4C}$  leads to

$$\frac{1}{4} \mathfrak{g}_2 \|\vec{u}\|_{m,2}^2 \leq \int_{\Omega} \varphi(x) + C \left( \frac{4C}{\mathfrak{g}_2} \right)^{r/(2-r)} - \mathfrak{F}(\vec{u}) + \langle \mathfrak{F}'(\vec{u}), \vec{u} \rangle \quad (5.2)$$

for any  $\vec{u} \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ . Let the sequence  $\{\vec{u}_k\}_{k \geq 1} \subset W_0^{m,2}(\Omega, \mathbb{R}^N)$  such that  $\sup_k |\mathfrak{F}(\vec{u}_k)| \leq M$  for some  $M > 0$ , and  $\mathfrak{F}'(\vec{u}_k) \rightarrow 0$  (resp.  $(1 + \|\vec{u}_k\|_{m,2})\mathfrak{F}'(\vec{u}_k) \rightarrow 0$ ). Then (5.2) implies (because of  $|\langle \mathfrak{F}'(\vec{u}), \vec{u} \rangle| \leq \|\mathfrak{F}'(\vec{u})\| \cdot \|\vec{u}\|_{m,2} \leq (1 + \|\vec{u}\|_{m,2})\|\mathfrak{F}'(\vec{u})\|$ ) that  $\{\vec{u}_k\}_{k \geq 1}$  is bounded in  $W_0^{m,2}(\Omega, \mathbb{R}^N)$  and thus has a convergent subsequence as above.  $\square$

## 6 Morse inequalities

Firstly, we show that Theorem 4.2 and Theorems 2.22, 2.23 imply the generalized Morse lemma.

**Theorem 6.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Sobolev domain,  $N \in \mathbb{N}$ , and  $H$  a closed subspace of  $W^{m,2}(\Omega, \mathbb{R}^N)$ . Let  $G$  be a compact Lie group which acts on  $H$  in a  $C^3$ -smooth isometric way. Suppose that Hypothesis  $\mathfrak{F}_{2,N}$  is satisfied and that the functional  $\mathfrak{F}$  given by (1.3) is  $G$ -invariant. Let  $\mathcal{O}$  be an isolated critical orbit of  $\mathfrak{F}$  (always understanding as  $\mathfrak{F}|_H$ ). It is a compact  $C^3$  submanifold, whose normal bundle  $N\mathcal{O}$  has fiber at  $\vec{u} \in \mathcal{O}$ ,*

$$N\mathcal{O}_{\vec{u}} = \{\vec{v} \in H \mid (\vec{v}, \vec{w})_{m,2} = 0 \ \forall \vec{w} \in T_{\vec{u}}\mathcal{O} \subset H\}.$$

Let  $N^+\mathcal{O}_{\vec{u}}$ ,  $N^0\mathcal{O}_{\vec{u}}$  and  $N^-\mathcal{O}_{\vec{u}}$  be the positive definite, null and negative definite spaces of the bounded linear self-adjoint operator associated with the bilinear form

$$N\mathcal{O}_{\vec{u}} \times N\mathcal{O}_{\vec{u}} \ni (\vec{v}, \vec{w}) \mapsto \sum_{i=1}^N \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}(x), \dots, D^m \vec{u}(x)) D^{\beta} v^j \cdot D^{\alpha} w^i dx.$$

Then  $\dim N^0\mathcal{O}_{\vec{u}}$  and  $\dim N^-\mathcal{O}_{\vec{u}}$  are finite and independent of choice of  $\vec{u} \in \mathcal{O}$ . They are called nullity and Morse index of  $\mathcal{O}$ , denoted by  $\nu_{\mathcal{O}}$  and  $\mu_{\mathcal{O}}$ , respectively. Moreover, the following holds.

- (i) If  $\nu_{\mathcal{O}} = 0$  (i.e., the critical orbit  $\mathcal{O}$  is nondegenerate), there exist  $\epsilon > 0$  and a  $G$ -equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers

$$\Phi : N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow N\mathcal{O}$$

such that for any  $\vec{u} \in \mathcal{O}$  and  $(\vec{v}_+, \vec{v}_-) \in N^+\mathcal{O}(\epsilon)_{\vec{u}} \times N^-\mathcal{O}(\epsilon)_{\vec{u}}$ ,

$$\mathfrak{F} \circ E \circ \Phi(\vec{u}, \vec{v}_+ + \vec{v}_-) = \|\vec{v}_+\|_{m,2}^2 - \|\vec{v}_-\|_{m,2}^2 + \mathfrak{F}|_{\mathcal{O}}, \quad (6.1)$$

where  $E : N\mathcal{O} \rightarrow H$  is given by  $E(\vec{u}, \vec{v}) = \vec{u} + \vec{v}$ .

- (ii) If  $\nu_{\mathcal{O}} \neq 0$  there exist  $\epsilon > 0$ , a  $G$ -equivariant topological bundle morphism that preserves the zero section,

$$\mathfrak{h} : N^0\mathcal{O}(3\epsilon) \rightarrow N^+\mathcal{O} \oplus N^-\mathcal{O} \subset H, \quad (\vec{u}, \vec{v}) \mapsto \mathfrak{h}_{\vec{u}}(\vec{v}),$$

and a  $G$ -equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers,  $\Phi : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \rightarrow N\mathcal{O}$ , such that the following properties hold:

- (ii.1) for any  $\vec{u} \in \mathcal{O}$  and  $(\vec{v}_0, \vec{v}_+, \vec{v}_-) \in N^0\mathcal{O}(\epsilon)_{\vec{u}} \times N^+\mathcal{O}(\epsilon)_{\vec{u}} \times N^-\mathcal{O}(\epsilon)_{\vec{u}}$ ,

$$\mathfrak{F} \circ E \circ \Phi(\vec{u}, \vec{v}_0, \vec{v}_+ + \vec{v}_-) = \|\vec{v}_+\|_{m,2}^2 - \|\vec{v}_-\|_{m,2}^2 + \mathfrak{F}(\vec{u} + \vec{v}_0 + \mathfrak{h}_{\vec{u}}(\vec{v}_0)); \quad (6.2)$$

- (ii.2) for each  $\vec{u} \in \mathcal{O}$  the function

$$N^0\mathcal{O}(\epsilon)_{\vec{u}} \rightarrow \mathbb{R}, \quad \vec{v} \mapsto \mathfrak{F}_{\vec{u}}^{\circ}(\vec{v}) := \mathfrak{F}(\vec{u} + \vec{v} + \mathfrak{h}_{\vec{u}}(\vec{v})) \quad (6.3)$$

is  $G_{\vec{u}}$ -invariant, of class  $C^1$ , and satisfies

$$D\mathfrak{F}_{\vec{u}}^{\circ}(\vec{v})\vec{w} := (\nabla \mathfrak{F}(\vec{u} + \vec{v} + \mathfrak{h}_{\vec{u}}(\vec{v})), \vec{w}), \quad \forall \vec{w} \in N^0\mathcal{O}_{\vec{u}}.$$

*Proof.* Since  $\mathfrak{F}|_H$  satisfies Hypothesis 1.1 with  $X = H$  around each critical point by Theorem 4.2, it follows from Lemma 2.10 that for each  $\vec{u} \in \mathcal{O}$  the restriction  $\mathfrak{F}|_{N\mathcal{O}_{\vec{u}}}$  satisfies Hypothesis 1.1 with  $X = N\mathcal{O}_{\vec{u}}$  around the origin of  $N\mathcal{O}_{\vec{u}}$ . Note that the exponential map on  $H$ ,  $\exp : TH = H \times H \rightarrow H$ , is given by  $\exp(\vec{u}, \vec{v}) = \vec{u} + \vec{v}$ . Then Theorems 2.22, 2.23 lead to the desired conclusions immediately.  $\square$

By Corollary 2.27 and (2.93), for any commutative ring  $\mathbf{K}$  we get

$$C_q(\mathfrak{F}, \mathcal{O}; \mathbf{K}) \cong \bigoplus_{j=0}^q C_{q-j-\mu_{\mathcal{O}}}(\mathfrak{F}_{\vec{u}}^{\circ}, \theta; \mathbf{K}) \otimes H_j(\mathcal{O}; \mathbf{K}) \quad \forall q = 0, 1, 2, \dots \quad (6.4)$$

if  $\vec{u} \in \mathcal{O}$ ,  $\nu_{\mathcal{O}} \neq 0$  and  $\mathcal{O}$  has trivial normal bundle; and  $C_*(\mathfrak{F}, \mathcal{O}; \mathbb{Z}_2) \cong C_{*-\mu_{\mathcal{O}}}(\mathcal{O}; \mathbb{Z}_2)$ ,

$$C_*(\mathfrak{F}, \mathcal{O}; \mathbf{K}) \cong C_{*-\mu_{\mathcal{O}}}(\mathcal{O}; \theta^- \otimes \mathbf{K}) \quad \text{and} \quad C_G^*(\mathfrak{F}, \mathcal{O}; \mathbf{K}) \cong H_G^{\mu_{\mathcal{O}}-1}(\mathcal{O}; \theta^- \otimes \mathbf{K}) \quad (6.5)$$

if  $\nu_{\mathcal{O}} = 0$ , where  $\mu_{\mathcal{O}}$  is the Morse index of  $\mathcal{O}$  and  $\theta^-$  is the orientation bundle of  $N^-\mathcal{O}$ .

From the second equality in (6.5) and [13, Chapter I, Theorem 7.6] we immediately arrive at

**Theorem 6.2.** *Under the assumptions of Theorem 6.1, Let  $a < b$  be two regular values of  $\mathfrak{F}$  and  $\mathfrak{F}^{-1}([a, b])$  contains only nondegenerate critical orbits  $\mathcal{O}_j$  with Morse indexes  $\mu_j$ ,  $j = 1, \dots, k$ . Suppose that  $\mathfrak{F}$  satisfies the  $(PS)_c$  condition for each  $c \in [a, b)$ . (For example, this is true if either  $\mathfrak{F}$  is coercive or one of Theorems 5.2, 5.3 holds in case  $H = W_0^{m,2}(\Omega, \mathbb{R}^N)$ .) Then there exists a polynomial with nonnegative integral coefficients  $Q(t)$  such that*

$$\sum_{i=0}^{\infty} \sum_{j=1}^k \text{rank} H_G^i(\mathcal{O}_j, \theta_j^- \otimes \mathbf{K}) t^{\mu_j+i} = \sum_{i=0}^{\infty} \text{rank} H_G^i(\mathfrak{F}_b, \mathfrak{F}_a; \mathbf{K}) t^i + (1+t)Q(t), \quad (6.6)$$

where  $\theta_j^-$  is the orientation bundle of  $N^- \mathcal{O}_j$ ,  $j = 1, \dots, k$ . In particular, if  $G$  is trivial and each  $\mathcal{O}_j$  becomes a nondegenerate critical point  $\vec{u}_j$ , then the following Morse inequalities hold:

$$\sum_{j=0}^l (-1)^{l-j} N_j(a, b) \geq \sum_{j=0}^l (-1)^{l-j} \beta_j(a, b), \quad l = 0, 1, \dots, \quad (6.7)$$

where for each  $q \in \mathbb{N} \cup \{0\}$ ,  $N_q(a, b) = \#\{1 \leq i \leq k \mid \mu_i = q\}$  (the number of points in  $\{\vec{u}_j\}_{j=1}^k$  with Morse index  $q$ ) and

$$\beta_q(a, b) = \sum_{i=1}^k \text{rank} H_q(\mathfrak{F}_b, \mathfrak{F}_a; \mathbf{K}).$$

Furthermore, if  $\mathfrak{F}$  is coercive, has only nondegenerate critical points, and for each  $q \in \{0\} \cup \mathbb{N}$  there exist only finitely many critical points with Morse index  $q$ , then the following relations hold:

$$\sum_{i=0}^q (-1)^{q-i} N_i \geq (-1)^q, \quad q = 0, 1, 2, \dots, \quad \text{and} \quad \sum_{i=0}^{\infty} (-1)^i N_i = 1, \quad (6.8)$$

where  $N_i$  is the number of critical points of  $\mathfrak{F}$  with Morse index  $i$ .

The proof of (6.8) is standard, see the proof of [4, Corollary 6.5.10]. When  $H = W_0^{m,2}(\Omega)$  and  $\mathfrak{F}$  is coercive, (6.7) was first obtained by Skrypnik in [58, §5.2].

## 7 Bifurcations for Quasi-linear elliptic systems

**Hypothesis 7.1.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded Sobolev domain,  $N \in \mathbb{N}$ , and functions

$$F : \overline{\Omega} \times \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)} \rightarrow \mathbb{R} \quad \text{and} \quad \mathbf{G} : \overline{\Omega} \times \prod_{k=0}^{m-1} \mathbb{R}^{N \times M_0(k)} \rightarrow \mathbb{R}$$

satisfy Hypothesis  $\mathfrak{F}_{p,N}$  and (i)-(ii) in Hypothesis  $\mathfrak{F}_{p,N}$ , respectively. Let  $V$  be a closed subspace of  $W^{m,p}(\Omega, \mathbb{R}^N)$  containing  $W_0^{m,p}(\Omega, \mathbb{R}^N)$ .

We consider (generalized) bifurcation solutions of the boundary value problem corresponding to the subspace  $V$ :

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha F_\alpha^i(x, \vec{u}, \dots, D^m \vec{u}) = \lambda \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} D^\alpha \mathbf{G}_\alpha^i(x, \vec{u}, \dots, D^{m-1} \vec{u}),$$

$$i = 1, \dots, N. \quad (7.1)$$

Call  $\vec{u} \in V$  a *generalized solution* of (7.1) if it is a critical point on  $V$  of the variational problem

$$\mathfrak{F}(\vec{u}) - \lambda \mathfrak{G}(\vec{u}) = \int_{\Omega} F(x, \vec{u}, \dots, D^m \vec{u}) dx - \lambda \int_{\Omega} \mathbf{G}(x, \vec{u}, \dots, D^{m-1} \vec{u}) dx. \quad (7.2)$$

As a generalization of [60, Theorem 7.2, Chapter 4], we may derive from Theorem 4.2 and Theorem 7.1 of [60, Chapter 4]:

**Theorem 7.2.** *Under Hypothesis 7.1, assume*

- (i) *the functionals  $\mathfrak{F}$  and  $\mathfrak{G}$  are even,  $\mathfrak{F}(\theta) = \mathfrak{G}(\theta) = 0$ ,  $\mathfrak{G}(\vec{u}) \neq 0 \forall \vec{u} \in V \setminus \{\theta\}$ , and  $\mathfrak{G}'(\vec{u}) \neq \theta \forall \vec{u} \in V \setminus \{\theta\}$ ;*
- (ii)  *$\langle \mathfrak{F}'(\vec{u}), \vec{u} \rangle \geq \nu(\|\vec{u}\|_{m,p})$ , where  $\nu(t)$  is a continuous function and positive for  $t > 0$ ;*
- (iii)  *$\mathfrak{F}(\vec{u}) \rightarrow +\infty$  as  $\|\vec{u}\|_{m,p} \rightarrow \infty$ .*

*Then for any  $c > 0$  there exists at least a sequence  $\{(\lambda_j, \vec{u}_j)\}_j \subset \mathbb{R} \times \{\vec{u} \in V \mid \mathfrak{F}(\vec{u}) = c\}$  satisfying (7.1).*

By Theorems 3.3, 4.2 we have

**Theorem 7.3.** *Under Hypothesis 7.1 with  $p = 2$ , let  $\vec{u}_0 \in V$  satisfy  $\mathfrak{F}'(\vec{u}_0) = 0$  and  $\mathfrak{G}'(\vec{u}_0) = 0$ . If  $(\lambda^*, \vec{u}_0)$  with certain  $\lambda^* \in \mathbb{R}$  is a bifurcation point for (7.1), then the linear problem*

$$\begin{aligned} & \sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m} (-1)^\alpha [F_{\alpha\beta}^{ij}(x, \vec{u}_0(x), \dots, D^m \vec{u}_0(x)) D^\beta v^j] \\ &= \lambda \sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m-1} (-1)^\alpha [\mathbf{G}_{\alpha\beta}^{ij}(x, \vec{u}_0(x), \dots, D^{m-1} \vec{u}_0(x)) D^\beta v^j] \end{aligned} \quad (7.3)$$

*with  $\lambda = \lambda^*$  has a nontrivial solution in  $V$ , i.e.,  $\vec{u}_0$  is a degenerate critical point of the functional  $\mathfrak{F} - \lambda^* \mathfrak{G}$  on  $V$ .*

Conversely, if  $\dim \Omega = 1$  and  $\vec{u}_0$  is a degenerate critical point of the functional  $\mathfrak{F} - \lambda^* \mathfrak{G}$ , using Theorem 3.4 we may obtain the corresponding bifurcation results. These will be given in more general forms, see [48].

**Hypothesis 7.4.** Let Hypothesis 7.1 hold with  $p = 2$ ,  $\vec{u}_0 \in V$  satisfy  $\mathfrak{F}'(\vec{u}_0) = 0$  and  $\mathfrak{G}'(\vec{u}_0) = 0$ , and the linear problem

$$\sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m} (-1)^\alpha [F_{\alpha\beta}^{ij}(x, \vec{u}_0(x), \dots, D^m \vec{u}_0(x)) D^\beta v^j] = 0 \quad (7.4)$$

have no nontrivial solutions in  $V$ .

The final condition in this hypothesis means that  $\mathfrak{F}''(\vec{u})$  has a bounded linear inverse. Under Hypothesis 7.4, by the arguments above Theorem 3.5, the all eigenvalues of (7.3) form a discrete subset of  $\mathbb{R}$ ,  $\{\lambda_j\}_{j=1}^\infty$ , which contains no zero and satisfies  $|\lambda_j| \rightarrow \infty$  as  $j \rightarrow \infty$ ; moreover, each  $\lambda_j$  has finite multiplicity. Let  $V_j$  be the eigensubspace of (7.3) corresponding to the eigenvalue  $\lambda_j$ ,  $j = 1, 2, \dots$ . By Theorems 3.5, 4.2 we directly obtain

**Theorem 7.5.** *Under Hypothesis 7.4, for an eigenvalue  $\lambda_k$  of (7.3) as above, assume that one of the following three conditions holds:*

(a)  $\mathfrak{F}''(\vec{u}_0)$  is positive definite, i.e., for each  $v \in V \setminus \{\theta\}$ ,

$$\sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}_0(x), \dots, D^m \vec{u}_0(x)) D^\beta v^j(x) D^\alpha v^i(x) dx > 0; \quad (7.5)$$

(b)  $\mathfrak{F}''(\vec{u}_0)$  is negative definite, i.e., for each  $v \in V \setminus \{\theta\}$ ,

$$\sum_{i,j=1}^N \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} F_{\alpha\beta}^{ij}(x, \vec{u}_0(x), \dots, D^m \vec{u}_0(x)) D^\beta v^j(x) D^\alpha v^i(x) dx < 0; \quad (7.6)$$

(c) each  $V_j$  is an invariant subspace of  $\mathfrak{F}''(\vec{u}_0)$  in  $V$ ,  $j = 1, 2, \dots$ , and either (7.5) holds for all  $v \in V_k \setminus \{\theta\}$ , or (7.6) does for all  $v \in V_k \setminus \{\theta\}$ .

Then  $(\lambda_k, \vec{u}_0) \in \mathbb{R} \times V$  is a bifurcation point for the problem (7.1), and one of the following alternatives occurs:

(i)  $(\lambda_k, \vec{u}_0)$  is not an isolated solution of (7.1) in  $\{\lambda_k\} \times V$ .

(ii) there exists a sequence  $\{\kappa_j\}_{j \geq 1} \subset \mathbb{R} \setminus \{\lambda_k\}$  such that  $\kappa_j \rightarrow \lambda_k$  and that for each  $\kappa_j$  the problem (7.1) with  $\lambda = \kappa_j$  has infinitely many solutions converging to  $\vec{u}_0 \in V$ .

(iii) for every  $\lambda$  in a small neighborhood of  $\lambda_k$  there is a nontrivial solution  $\vec{u}_\lambda$  of (7.1) converging to  $\vec{u}_0$  as  $\lambda \rightarrow \lambda_k$ ;

(iv) there is a one-sided  $\Lambda$  neighborhood of  $\lambda_k$  such that for any  $\lambda \in \Lambda \setminus \{\lambda_k\}$ , (7.1) has at least two nontrivial solutions converging to  $\vec{u}_0$  as  $\lambda \rightarrow \lambda_k$ .

**Remark 7.6.** (i) When  $N = 1$ ,  $V = W_0^{m,2}(\Omega)$ ,  $u = \theta$  and  $\mathfrak{F}$  also satisfies

$$(\mathfrak{F}'(u), u)_{m,2} \geq c \|u\|_{m,2}^2 \quad (7.7)$$

for some  $c > 0$  and all sufficiently small  $\|u\|_{m,2}$ , if  $\lambda^*$  is an eigenvalue of (7.3) with  $u = \theta$ , it was proved in [59, Chap.1, Theorem 3.5] that  $(\lambda, \theta)$  is a bifurcation point of (7.1). Since  $\mathfrak{F}(\theta) = \theta$ , it is clear that (7.7) implies  $\mathfrak{F}''(\theta)$  to be positive definite. Hence Theorem 7.5 contains [59, Chap.1,

Theorem 3.5] as a special example.

(ii) When  $N = 1$ ,  $n \geq 3$ ,  $V = H_0^1(\Omega)$ ,  $\mathbf{G}(x, \xi_0, \dots, \xi_n) = \frac{1}{2}\xi_0^2$  and

$$F(x, \xi_0, \dots, \xi_n) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x, \xi_0) \xi_i \xi_j - \int_0^{\xi_0} g(x, t) dt, \quad (7.8)$$

Canino [11, Theorem 1.3] obtained a corresponding result provided that functions  $a_{ij} = a_{ji}$ ,  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following assumptions:

a.0)  $a_{ij}$  is of class  $C^1$ , and is of class  $C^2$  in  $\xi_0$  for a.e.  $x \in \Omega$ ;

a.1) there exists  $C > 0$  such that for a.e.  $x \in \Omega$ , for all  $\xi_0 \in \mathbb{R}$  and for all  $i, j, k$ ,

$$\begin{aligned} |a_{ij}(x, \xi_0)| &\leq C, & |D_{\xi_0} a_{ij}(x, \xi_0)| &\leq C, \\ |D_{x_k} a_{ij}(x, \xi_0)| &\leq C, & |D_{\xi_0 \xi_0}^2 a_{ij}(x, \xi_0)| &\leq C; \end{aligned}$$

a.2) there exists  $\nu > 0$  such that for a.e.  $x \in \Omega$ , for all  $\xi_i \in \mathbb{R}$ ,  $i = 0, \dots, n$ ,

$$\sum_{i,j=1}^n a_{ij}(x, \xi_0) \xi_i \xi_j \geq \nu \sum_{i=1}^n \xi_i^2;$$

a.3) for a.e.  $x \in \Omega$ , for all  $\xi_i \in \mathbb{R}$ ,  $i = 0, \dots, n$ ,

$$\sum_{i,j=1}^n s D_{\xi_0} a_{ij}(x, \xi_0) \xi_i \xi_j \geq 0;$$

g) for every  $\xi_0 \in \mathbb{R}$ ,  $g(x, \xi_0)$  is measurable with respect to  $x$ , for a.e.  $x \in \Omega$ ,  $g(x, \xi_0)$  is of class  $C^1$  with respect to  $s$ ,  $g(x, 0) = 0$ ; moreover, there exist  $b \in \mathbb{R}$  and  $0 < p < 4/(n-2)$  such that, for a.e.  $x \in \Omega$  and all  $s \in \mathbb{R}$ ,

$$|D_{\xi_0} g(x, \xi_0)| \leq b(1 + |\xi_0|^p).$$

It is easily checked that  $F$  in (7.8) satisfies Hypothesis  $\mathfrak{F}_{2,1}$ . Thus [11, Theorem 1.3] is implied in Theorem 7.5 with  $N = 1$  and  $\vec{u} = \theta$ .

By Theorems 3.7, 3.9 we deduce

**Theorem 7.7.** *Under Hypothesis 7.1 with  $p = 2$ , let  $G$  be a compact Lie group acting on  $V$  in a  $C^3$ -smooth and isometric (and so orthogonal) way. Suppose that both  $\mathfrak{F}$  and  $\mathfrak{G}$  are  $G$ -invariant, and that  $\vec{u}_0 \in \text{Fix}(G)$  satisfies  $\mathfrak{F}'(\vec{u}_0) = 0$  and  $\mathfrak{G}'(\vec{u}_0) = 0$ . For an eigenvalue  $\lambda_k$  of (7.3) as above, assume that one of the three conditions (a), (b) and (c) in Theorem 7.5 holds. Then  $(\lambda_k, \vec{u}_0) \in \mathbb{R} \times V$  is a bifurcation point for the equation (3.14), and if  $\dim V_k \geq 2$  and the unit sphere in  $V_k$  is not a  $G$ -orbit we must get one of the following alternatives:*

- (i)  $(\lambda_k, \vec{u}_0)$  is not an isolated solution of (7.1) in  $\{\lambda_k\} \times V$ ;
- (ii) there exists a sequence  $\{\kappa_j\}_{j \geq 1} \subset \mathbb{R} \setminus \{\lambda_k\}$  such that  $\kappa_j \rightarrow \lambda_k$  and that for each  $\kappa_j$  the problem (7.1) with  $\lambda = \kappa_j$  has infinitely many  $G$ -orbits of solutions converging to  $\vec{u}_0 \in V$ ;

(iii) for every  $\lambda$  in a small neighborhood of  $\lambda_k$  there is a nontrivial solution  $\vec{u}_\lambda$  of (7.1) converging to  $\vec{u}_0$  as  $\lambda \rightarrow \lambda_k$ ;

(iv) there is a one-sided  $\Lambda$  neighborhood of  $\lambda_k$  such that for any  $\lambda \in \Lambda \setminus \{\lambda_k\}$ , (7.1) has at least two nontrivial critical orbits converging to  $\vec{u}_0$  as  $\lambda \rightarrow \lambda_k$ .

Furthermore, if the Lie group  $G$  is equal to  $\mathbb{Z}_2$  or  $S^1$ , then the above (iii)-(iv) can be replaced by the following

(iii') there exist left and right neighborhoods  $\Lambda^-$  and  $\Lambda^+$  of  $\lambda_k$  in  $\mathbb{R}$  and integers  $n^+, n^- \geq 0$ , such that  $n^+ + n^- \geq \dim V$  and for  $\lambda \in \Lambda^- \setminus \{\lambda^*\}$  (resp.  $\lambda \in \Lambda^+ \setminus \{\lambda^*\}$ ), (7.1) has at least  $n^-$  (resp.  $n^+$ ) distinct critical  $G$ -orbits different from  $\vec{u}_0$ , which converge to  $\vec{u}_0$  as  $\lambda \rightarrow \lambda_k$ .

The corresponding claims to 3) of Theorem 3.7 can be easily written.

If  $\vec{u}_0 \notin \text{Fix}(G)$ , Theorem 3.20 may yield a result. If  $n = \dim \Omega = 1$ , by Theorems 3.14, 3.16, 3.17 we can also obtain more results. They will be given in [46, 48].

### Example

For  $0 < T_i < \infty$ ,  $i = 1, \dots, n$ , and  $\Omega := \prod_{i=1}^n (0, T_i)$  let  $C^m(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$  be the set of all  $\vec{u} \in C^m(\mathbb{R}^n, \mathbb{R}^N)$  which are  $T_i$ -periodic with respect to the  $i$ -th variable,  $i = 1, \dots, n$ . It may be viewed as a subspace of  $W^{m,2}(\Omega, \mathbb{R}^N)$ . Denote by  $W^{m,2}(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$  the closure of  $C^m(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$  in  $W^{m,2}(\Omega, \mathbb{R}^N)$ . Clearly,  $W_0^{m,2}(\Omega, \mathbb{R}^N)$  is contained in  $W^{m,2}(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$ . The compact connected Lie group  $G = \prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z})$ , which is isomorphic to  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ , acts on  $W^{m,2}(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$  via the following isometric linear representation:

$$([t_1, \dots, t_n] \cdot \vec{u})(x_1, \dots, x_n) = (u_1(x_1 + t_1), \dots, u_n(x_n + t_n)) \quad (7.9)$$

for  $[t_1, \dots, t_n] \in G$  and  $\vec{u} = (u_1, \dots, u_n) \in W^{m,2}(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$ . The set of fixed points of this action,  $\text{Fix}(G)$ , consist of all constant vector functions from  $\bar{\Omega}$  to  $\mathbb{R}^N$ . Under Hypothesis 7.1 with  $\Omega = \prod_{i=1}^n (0, T_i) \subset \mathbb{R}^n$ , assume also that  $F(x, \xi)$  and  $\mathbf{G}(x, \xi)$  satisfy

$$\begin{aligned} F(x_1, \dots, x_{\hat{i}}, \dots, x_n, \xi) &= F(x_1, \dots, x_{\hat{i}}, \dots, x_n, \xi), \\ \mathbf{G}(x_1, \dots, x_{\hat{i}}, \dots, x_n, \xi) &= \mathbf{G}(x_1, \dots, x_{\hat{i}}, \dots, x_n, \xi), \end{aligned}$$

where  $x_{\hat{i}} = 0$  and  $x_{\hat{i}} = T_i$ ,  $i = 1, \dots, n$ . In other words,  $F$  and  $\mathbf{G}$  may be viewed as functions on  $\mathbb{R}^n \times \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)}$  with period  $T_i$  in variables  $x_i$ ,  $i = 1, \dots, n$ . Then the functionals  $\mathfrak{F}$  and  $\mathfrak{G}$  are  $G$ -invariant, and every critical orbit different from points in  $\text{Fix}(G)$  must be homeomorphic to some  $T^s$ ,  $1 \leq s \leq n$ . Clearly, if some  $\vec{u} \in W^{m,2}(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$  are constant with respect to variables  $x_{i_r}$ ,  $r = 1, \dots, k < n$ , but not with respect to any other variable  $x_i$ , then the orbit  $G(\vec{u})$  of  $\vec{u}$  is homeomorphic to some  $T^k$ . However, it is possible that the orbit of  $\vec{u}$  is of  $T^k$ -type even if  $\vec{u}$  is not constant with respect to each variable, see the proof of [65, Proposition 3.4].



Clearly, Hypothesis 7.4 is satisfied for each constant map  $\vec{u}_0 : \bar{\Omega} \rightarrow \mathbb{R}^N$ . So Theorem 7.5, and Theorem 7.7 with  $G = T^n$  can be directly applied. By Theorem 3.7 we also get

**Theorem 7.8.** *Let  $\vec{u}_0 : \bar{\Omega} \rightarrow \mathbb{R}^N$  be a constant map. Assume that  $F$  and  $G$  satisfy the conditions of Theorem 7.5 with  $V = W^{m,2}(\prod_{i=1}^n \mathbb{R}/(T_i\mathbb{Z}), \mathbb{R}^N)$ . Then  $(\lambda_k, \vec{u}_0) \in \mathbb{R} \times V$  is a bifurcation point for the equation (7.1). Moreover, if  $V_k$  (the eigensubspace of (7.3) corresponding to the eigenvalue  $\lambda_k$ ) satisfies one of the following assumptions: A)  $\text{Fix}(T^n) \cap V_k = \{\theta\}$ , B) every orbit in  $V_k$  is homeomorphic to some  $T^s$  for  $s \geq 2$ ; then either one of the above (i)-(iii) in Theorem 7.7 with  $G = T^n$  or the following hold:*

**(iv)'** *there is a one-sided  $\Lambda$  neighborhood of  $\lambda_k$  such that for any  $\lambda \in \Lambda \setminus \{\lambda_k\}$ , the problem (7.1) with  $\lambda = \lambda_k$  has at least  $\dim V_k$  (resp.  $2 \dim V_k$ ) nontrivial critical orbits in the case A) (resp. B)), where every orbit is counted with its multiplicity (defined by [68, Definition 1.3]).*

The theory in this paper provides necessary tools for generalizing [39, 65] to the functional considered in the above example. They will be investigated in the latter paper.

Finally, we present a result associated with Theorems 5.4.2, 5.7.4 in [26].

**Theorem 7.9.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Sobolev domain,  $N \in \mathbb{N}$ . Suppose that*

$$\bar{\Omega} \times \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)} \times [0, 1] \ni (x, \xi, \lambda) \mapsto F(x, \xi; \lambda) \in \mathbb{R}$$

*is differentiable with respect to  $\lambda$ , and satisfies the following conditions:*

- (i)** *All  $F(\cdot; \lambda)$  satisfy Hypothesis  $\mathfrak{F}_{2,N}$  uniformly with respect to  $\lambda \in [0, 1]$ , i.e., the inequalities (1.1) and (1.2) are uniformly satisfied for all  $\lambda \in [0, 1]$ .*
- (ii)** *If  $q_\alpha = 1$  for  $|\alpha| < m - n/2$ , and  $q_\alpha = 2_\alpha/(2_\alpha - 1)$  for  $m - n/2 \leq |\alpha| \leq m$ , then*

$$\sup_{|\alpha| \leq m} \sup_{1 \leq i \leq N} \sup_{\lambda} \int_{\Omega} [|D_\lambda F(x, 0; \lambda)| + |D_\lambda F_\alpha^i(x, 0; \lambda)|^{q_\alpha}] dx < \infty.$$

- (iii)** *For all  $i = 1, \dots, N$  and  $|\alpha| \leq m$ ,*

$$\begin{aligned} & |D_\lambda F_\alpha^i(x, \xi; \lambda)| \leq |D_\lambda F_\alpha^i(x, 0; \lambda)| \\ & + \mathfrak{g}\left(\sum_{k=1}^N |\xi_0^k|\right) \sum_{|\beta| < m-n/2} \left(1 + \sum_{k=1}^N \sum_{m-n/2 \leq |\gamma| \leq m} |\xi_\gamma^k|^{2_\gamma}\right)^{2_{\alpha\beta}} \\ & + \mathfrak{g}\left(\sum_{k=1}^N |\xi_0^k|\right) \sum_{l=1}^N \sum_{m-n/2 \leq |\beta| \leq m} \left(1 + \sum_{k=1}^N \sum_{m-n/2 \leq |\gamma| \leq m} |\xi_\gamma^k|^{2_\gamma}\right)^{2_{\alpha\beta}} |\xi_\beta^l|; \end{aligned}$$

*where  $\mathfrak{g} : [0, \infty) \rightarrow \mathbb{R}$  is a continuous, positive, nondecreasing function.*

*Let  $V$  be a closed subspace of  $W^{m,2}(\Omega, \mathbb{R}^N)$ , and for each  $\lambda \in [0, 1]$  let  $\vec{u}_\lambda$  be a critical point of the functional*

$$\mathfrak{F}_\lambda(\vec{u}) = \int_{\Omega} F(x, \vec{u}, \dots, D^m \vec{u}; \lambda) dx$$

on  $V$ . Suppose that  $[0, 1] \ni \lambda \mapsto \vec{u}_\lambda \in V$  is continuous. Then one of the following alternatives occurs:

- (I) There exists certain  $\lambda_0 \in [0, 1]$  such that  $(\lambda_0, \vec{u}_{\lambda_0})$  is a bifurcation point of  $\nabla \mathfrak{F}_\lambda(\vec{u}) = 0$ .
- (II) Each  $\vec{u}_\lambda$  is an isolated critical point of  $\mathfrak{F}_\lambda$  and  $C_*(\mathfrak{F}_\lambda, \vec{u}_\lambda; \mathbf{K}) = C_*(\mathfrak{F}_0, \vec{u}_0; \mathbf{K})$  for all  $\lambda \in [0, 1]$ ; moreover  $\vec{u}_\lambda$  is a local minimizer of  $\mathfrak{F}_\lambda$  if and only if  $\vec{u}_0$  is a local minimizer of  $\mathfrak{F}_0$ .

If  $D_\lambda F(\cdot; \lambda)$  uniformly satisfy the inequalities (1.1) and (1.2) for all  $\lambda \in [0, 1]$ . Then these and (ii) can yield (iii).

*Proof.* Suppose that (I) does not hold. Then each  $\vec{u}_\lambda$  is an isolated critical point of  $\mathfrak{F}_\lambda$ . Since  $[0, 1] \ni \lambda \mapsto \vec{u}_\lambda \in V$  is continuous, we may find a bounded open subset  $\mathcal{O}$  in  $V$  such that  $\vec{u}_\lambda$  is a unique critical point of  $\mathfrak{F}_\lambda$  contained in the closure  $\overline{\mathcal{O}}$  of  $\mathcal{O}$ . Take  $R > 0$  such that  $\overline{\mathcal{O}} \subset B_V(\theta, R)$ .

As in the proof of (4.8) we may derive from (iii) that with  $q_\alpha$  in (ii),

$$\begin{aligned} |D_\lambda F(x, \xi; \lambda)| &\leq |D_\lambda F(x, 0; \lambda)| + \left( \sum_{k=1}^N |\xi_0^k| \right) \sum_{i=1}^N \sum_{|\alpha| < m-n/2} |D_\lambda F_\alpha^i(x, 0; \lambda)| \\ &+ \sum_{i=1}^N \sum_{m-n/2 \leq |\alpha| \leq m} |D_\lambda F_\alpha^i(x, 0; \lambda)|^{q_\alpha} + \widehat{\mathfrak{g}} \left( \sum_{k=1}^N |\xi_0^k| \right) \left( 1 + \sum_{l=1}^N \sum_{m-n/2 \leq |\alpha| \leq m} |\xi_\alpha^l|^{2\alpha} \right) \end{aligned}$$

for all  $(x, \xi, \lambda)$  and some continuous, positive, nondecreasing function  $\widehat{\mathfrak{g}} : [0, \infty) \rightarrow \mathbb{R}$ . As before we have a constant  $C = C(m, n, N, R) > 0$  such that

$$\sup \left\{ \sum_{|\alpha| < m-n/2} |D^\alpha \vec{u}(x)| \mid x \in \Omega \right\} < C \quad \text{for all } \vec{u} \in W^{m,2}(\Omega, \mathbb{R}^N) \text{ with } \|\vec{u}\|_{m,2} \leq R.$$

It follows that for any  $\lambda_i \in [0, 1]$ ,  $i = 1, 2$ ,

$$\begin{aligned} |\mathfrak{F}_{\lambda_1}(\vec{u}) - \mathfrak{F}_{\lambda_2}(\vec{u})| &\leq |\lambda_2 - \lambda_1| \int_\Omega \sup_\lambda |D_\lambda F(x, \vec{u}, \dots, D^m \vec{u}; \lambda)| dx \\ &\leq |\lambda_2 - \lambda_1| \left[ \sup_\lambda \int_\Omega |D_\lambda F(x, 0; \lambda)| dx + C \sum_{i=1}^N \sum_{|\alpha| < m-n/2} \sup_\lambda \int_\Omega |D_\lambda F_\alpha^i(x, 0; \lambda)| dx \right. \\ &+ \sum_{i=1}^N \sum_{m-n/2 \leq |\alpha| \leq m} \sup_\lambda \int_\Omega |D_\lambda F_\alpha^i(x, 0; \lambda)|^{q_\alpha} dx \\ &\left. + \widehat{\mathfrak{g}}(C) \int_\Omega \left( 1 + \sum_{l=1}^N \sum_{m-n/2 \leq |\alpha| \leq m} |D^\alpha u^l|^{2\alpha} \right) dx \right]. \end{aligned}$$

This implies that  $[0, 1] \ni \lambda \mapsto \mathfrak{F}_\lambda$  is continuous in  $C^0(\bar{B}_V(\theta, R))$ . Similarly, (ii)–(iii) yield the continuity of the map  $[0, 1] \ni \lambda \mapsto \nabla \mathfrak{F}_\lambda$  in  $C^0(\bar{B}_V(\theta, R), V)$ . Hence the map  $[0, 1] \ni \lambda \mapsto \mathfrak{F}_\lambda$  is continuous in  $C^1(\bar{B}_V(\theta, R))$ . As in Step 1 of the proof of Theorem 3.1, the stability of critical groups (cf. [16, Theorem III.4] and [22, Theorem 5.1]) leads to the first claim in (II).

For the second claim, it suffices to prove that  $\vec{u}_0$  is a local minimizer of  $\mathfrak{F}_0$  provided  $\vec{u}_\lambda$  is a local minimizer of  $\mathfrak{F}_\lambda$ . Since  $\vec{u}_\lambda$  is an isolated critical point of  $\mathfrak{F}_\lambda$ , by Example 1 in [13, page 33] we have  $C_q(\mathfrak{F}_\lambda, \vec{u}_\lambda; \mathbf{K}) = \delta_{q0}\mathbf{K}$  for  $q = 0, 1, \dots$ . It follows that  $C_q(\mathfrak{F}_0, \vec{u}_0; \mathbf{K}) = \delta_{q0}\mathbf{K}$  for  $q = 0, 1, \dots$ . By Theorem 2.3, this means that the Morse index of  $\mathfrak{F}_0$  at  $\vec{u}_0$  must be zero. We can assume  $\vec{u}_0 = \theta$  after replacing  $\mathfrak{F}_0$  by  $\mathfrak{F}_0(\vec{u}_0 + \cdot)$ . So  $C_q(\mathfrak{F}_0^\circ, \theta; \mathbf{K}) = \delta_{q0}\mathbf{K}$  for  $q = 0, 1, \dots$ . Then  $\theta$  is a local minimizer of  $\mathfrak{F}_0^\circ$  by Example 4 in [13, page 43]. It follows from Theorem 2.2 (or Theorem 6.1 with  $\mathcal{O} = \theta$ ) that  $\vec{u}_0 = \theta$  must be a local minimizer of  $\mathfrak{F}_0$ .  $\square$

## 8 Concluding remarks

In Section 3 we only generalize some bifurcation theorems for potential operators with the splitting theorem obtained in this paper. Once some splitting theorems are proved, the same ideas can be used to generalize some past bifurcation theorems. For example, we may obtain corresponding extended versions of [2, 1] in the variational frames of [39, 40] and [5, 34]. These and applications will be given in [46].

We here do not consider easy generalizations of the contents in Part II to a larger framework as in [55, 61, 54] because they are developed in a more general setting as in [35, 36], see [47]. Moreover, both the theory in Part I and that of [39, 40] are applicable to one-dimensional variational problem of higher order, see [48].

## A Proof of Proposition 4.3

Recall that we have written  $\xi \in \mathbb{R}^{M(m)}$  as  $\xi = \{\xi_\alpha : |\alpha| \leq m\}$  and denote by  $\xi_\circ = \{\xi_\alpha : |\alpha| < m - n/p\}$ . By the mean value theorem and (1.5) we get a collect of numbers  $\{t_\beta \in (0, 1) : |\beta| \leq m\}$  such that

$$\begin{aligned}
|f_\alpha(x, \xi)| - |f_\alpha(x, 0)| &\leq \sum_{|\beta| \leq m} |f_{\alpha\beta}(x, t_\beta \xi)| \cdot |\xi_\beta| \\
&\leq \sum_{|\beta| \leq m} \mathfrak{g}_1(|t_\beta \xi_\circ|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |t_\beta \xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \\
&\leq \sum_{|\beta| \leq m} \mathfrak{g}_1(|\xi_\circ|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \\
&\leq \sum_{|\beta| \leq m} \mathfrak{g}_1(|\xi_\circ|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta|
\end{aligned}$$

$$\begin{aligned}
&= \sum_{|\beta| < m-n/p} \mathfrak{g}_1(|\xi_\circ|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \\
&\quad + \sum_{m-n/p \leq |\beta| \leq m} \mathfrak{g}_1(|\xi_\circ|) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \\
&\leq \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|f_\alpha(x, \xi)| &\leq |f_\alpha(x, 0)| + \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta|, \tag{A.1}
\end{aligned}$$

which lead to (4.9) with  $\mathfrak{g}_4(|\xi_\circ|) := \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| + \mathfrak{g}_1(|\xi_\circ|)$ .

Suppose  $|\alpha| < m - n/p$ . Then  $p_{\alpha\beta} = 1 - 1/p_\beta = 1/q_\beta$  if  $m - n/p \leq |\beta| \leq m$ , and hence the second and third terms in (A.1), respectively, becomes

$$\begin{aligned}
&\mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\beta} \\
&\leq \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| M(m) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right), \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{g}_1(|\xi_\circ|) \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\beta} |\xi_\beta| \\
&\leq \mathfrak{g}_1(|\xi_\circ|) \sum_{m-n/p \leq |\beta| \leq m} \left[ \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right) + |\xi_\beta|^{p_\beta} \right] \\
&\leq \mathfrak{g}_1(|\xi_\circ|) (M(m) + 1) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right). \tag{A.3}
\end{aligned}$$

These and (A.1)-(A.2) give rise to (4.10) with  $\mathfrak{g}_5(|\xi_\circ|) := (M(m) + 1) \mathfrak{g}_1(|\xi_\circ|) (|\xi_\circ| + 1)$ .

Suppose  $m - n/p \leq |\alpha| \leq m$ . Then  $0 < p_{\alpha\beta} \leq 1 - 1/p_\alpha - 1/p_\beta$  if  $m - n/p \leq |\beta| \leq m$ . In

this case the second and third terms in (A.1), respectively, becomes

$$\begin{aligned}
& \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ| \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} \\
& \leq \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ| \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha} \\
& \leq \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ| M(m) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma/q_\alpha} \right)
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
& \mathfrak{g}_1(|\xi_\circ|) \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \\
& \leq \mathfrak{g}_1(|\xi_\circ|) \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha - 1/p_\beta} |\xi_\beta| \\
& \leq \mathfrak{g}_1(|\xi_\circ|) \sum_{|\beta| < m-n/p} \left[ \left( \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha - 1/p_\beta} \right)^{p_\beta/(p_\beta - q_\alpha)} + |\xi_\beta|^{p_\beta/q_\alpha} \right] \\
& \leq \mathfrak{g}_1(|\xi_\circ|) \sum_{|\beta| < m-n/p} \left[ \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha} + |\xi_\beta|^{p_\beta/q_\alpha} \right] \\
& \leq \mathfrak{g}_1(|\xi_\circ|)(M(m) + 1) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha}.
\end{aligned} \tag{A.5}$$

These lead to (4.11).

With a similar argument to (A.1) we obtain

$$\begin{aligned}
& |f(x, \xi)| - |f(x, 0)| \leq \sum_{|\alpha| \leq m} |f_\alpha(x, s_\alpha \xi)| \cdot |\xi_\alpha| \leq \sum_{|\alpha| \leq m} |f_\alpha(x, 0)| \cdot |\xi_\alpha| \\
& + \sum_{|\alpha| \leq m} \mathfrak{g}_1(|s_\alpha \xi_\circ|)|s_\alpha \xi_\circ| \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |s_\alpha \xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\alpha| \\
& + \sum_{|\alpha| \leq m} \mathfrak{g}_1(|s_\alpha \xi_\circ|) \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |s_\alpha \xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |s_\alpha \xi_\beta| \cdot |\xi_\alpha| \\
& \leq \sum_{|\alpha| \leq m} |f_\alpha(x, 0)| \cdot |\xi_\alpha| \\
& + \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ| \sum_{|\alpha| \leq m} \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\alpha| \\
& + \mathfrak{g}_1(|\xi_\circ|) \sum_{|\alpha| \leq m} \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \cdot |\xi_\alpha| \\
& = T_1 + T_2 + T_3.
\end{aligned}$$

We can estimate these three terms as follows.

$$\begin{aligned}
T_1 &= \sum_{|\alpha| < m-n/p} |f_\alpha(x, 0)| \cdot |\xi_\alpha| + \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, 0)| \cdot |\xi_\alpha| \\
&= |\xi_\circ| \sum_{|\alpha| < m-n/p} |f_\alpha(x, 0)| + \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, 0)| \cdot |\xi_\alpha| \\
&\leq |\xi_\circ| \sum_{|\alpha| < m-n/p} |f_\alpha(x, 0)| + \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, 0)|^{q_\alpha} + \sum_{m-n/p \leq |\alpha| \leq m} |\xi_\alpha|^{p_\alpha};
\end{aligned}$$

$$\begin{aligned}
T_2 &= \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| \sum_{|\alpha| < m-n/p} \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\alpha| \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| \sum_{m-n/p \leq |\alpha| \leq m} \sum_{|\beta| < m-n/p} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\alpha| \\
&\leq \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ|^2 M(m-n/p+1) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right) \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| M(m-n/p+1) \sum_{m-n/p \leq |\alpha| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{1/q_\alpha} |\xi_\alpha| \\
&\leq \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ|^2 M(m-n/p+1) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right) \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| M(m-n/p+1) \sum_{m-n/p \leq |\alpha| \leq m} \left[ \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right) + |\xi_\alpha|^{p_\alpha} \right] \\
&\leq \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ|^2 M(m) \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right) \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) |\xi_\circ| (M(m)+1)^2 \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right) \\
&= \mathfrak{g}_1(|\xi_\circ|) [|\xi_\circ|^2 M(m) + |\xi_\circ| (M(m)+1)^2] \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right);
\end{aligned}$$

$$\begin{aligned}
T_3 &= \mathfrak{g}_1(|\xi_\circ|) \sum_{|\alpha| < m-n/p} \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \cdot |\xi_\alpha| \\
&\quad + \mathfrak{g}_1(|\xi_\circ|) \sum_{m-n/p \leq |\alpha| \leq m} \sum_{m-n/p \leq |\beta| \leq m} \left( 1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma} \right)^{p_{\alpha\beta}} |\xi_\beta| \cdot |\xi_\alpha|
\end{aligned}$$

$$\begin{aligned}
&\leq \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ| \sum_{m-n/p \leq |\beta| \leq m} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right)^{1/q_\beta} |\xi_\beta| \\
&+ \mathfrak{g}_1(|\xi_\circ|) \sum_{m-n/p \leq |\alpha| \leq m} \sum_{m-n/p \leq |\beta| \leq m} \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right)^{p_{\alpha\beta}} |\xi_\beta| \cdot |\xi_\alpha| \\
&\leq \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ| \sum_{m-n/p \leq |\beta| \leq m} \left[ \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right) + |\xi_\beta|^{p_\beta} \right] \\
&+ \mathfrak{g}_1(|\xi_\circ|) \sum_{m-n/p \leq |\alpha| \leq m} \sum_{m-n/p \leq |\beta| \leq m} \left[ |\xi_\beta|^{p_\beta} + |\xi_\alpha|^{p_\alpha} \right. \\
&\quad \left. + \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right)^{p_{\alpha\beta}(1-p_\alpha^{-1}-p_\beta^{-1})} \right] \\
&\leq \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ|(M(m)+1) \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right) \\
&\quad + \mathfrak{g}_1(|\xi_\circ|)(M(m)+1)^2 \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right)
\end{aligned}$$

because  $0 < p_{\alpha\beta} < 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta}$  for  $m - n/p \leq |\alpha| \leq m$  and  $m - n/p \leq |\beta| \leq m$ .

In summary we get

$$\begin{aligned}
|f(x, \xi)| &\leq |f(x, 0)| + |\xi_\circ| \sum_{|\alpha| < m-n/p} |f_\alpha(x, 0)| + \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, 0)|^{q_\alpha} \\
&+ \sum_{m-n/p \leq |\alpha| \leq m} |\xi_\alpha|^{p_\alpha} \\
&+ \mathfrak{g}_1(|\xi_\circ|)[|\xi_\circ|^2 M(m) + |\xi_\circ|(M(m)+1)^2] \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right) \\
&+ \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ|(M(m)+1) \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right) \\
&+ \mathfrak{g}_1(|\xi_\circ|)(M(m)+1)^2 \left(1 + \sum_{m-n/p \leq |\gamma| \leq m} |\xi_\gamma|^{p_\gamma}\right) \\
&\leq |f(x, 0)| + |\xi_\circ| \sum_{|\alpha| < m-n/p} |f_\alpha(x, 0)| + \sum_{m-n/p \leq |\alpha| \leq m} |f_\alpha(x, 0)|^{q_\alpha} \\
&+ \mathfrak{g}_3(|\xi_\circ|) \left(1 + \sum_{m-n/p \leq |\alpha| \leq m} |\xi_\alpha|^{p_\alpha}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{g}_3(|\xi_\circ|) &= 1 + \mathfrak{g}_1(|\xi_\circ|)[|\xi_\circ|^2 M(m) + |\xi_\circ|(M(m)+1)^2] \\
&+ \mathfrak{g}_1(|\xi_\circ|)|\xi_\circ|(M(m)+1) + \mathfrak{g}_1(|\xi_\circ|)(M(m)+1)^2.
\end{aligned}$$

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